University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 3

Posted October 13, 2013 **Due Monday, October 22, 2013**

1. **It's not a tensor!** [from Brandenberger]

   This problem is a simple game. Identify which of the following equations could be valid tensor equations; for the ones that can't be, say why not. Here I mean tensors *e.g.* under the Lorentz group (or maybe some more general transformations) where we must distinguish between covariant (lower) and contravariant (upper) indices.

   (a) $R^a_{man} = T_{mn}$

   (b) $\text{☺}_a \omega_{bc} = \text{♃}_{ab}$

   (c) $\Diamond_{a\aleph\aleph} = \text{♄}_a$

   (d) $\text{♫}_{ab} + \text{☼}_{ac} = \Upsilon_{bc}$

   (e) $\natural_a(\sharp_b + \flat_b) = \text{☻}_{ab}$

   (f) $\text{☹}_a \bigstar_b = \text{☽}_{ab}$

2. **Relativistic charged particle**

   Consider the action of a (relativistic) charged particle coupled to a background gauge field, A_μ, living in Minkowski space. It is governed by the action

$$S[x^\mu] = \int \mathrm{d}\mathfrak{s} \, \left(-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - \frac{e}{c} A_\mu(x(\mathfrak{s}))\dot{x}^\mu(\mathfrak{s}). \right)$$

   where $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\mathfrak{s}}$.

   (a) **General covariance in one dimension.**
   Suppose we reparametrize the worldline according to

$$\mathfrak{s} \mapsto \tilde{\mathfrak{s}}(\mathfrak{s}).$$

   Use the chain rule to show that this change of coordinates preserves S.

   (b) Vary with respect to $x^\mu(\mathfrak{s})$ and find the Lorentz force law.

   (c) Vary with respect to $A_\mu(x)$ and find the source for Maxwell's equations produced by a trajectory of this particle.

3. Show that the current obtained from $j_\mu = \frac{\delta S}{\delta A}$ is conserved as long as the worldlines do not end.

4. **Non-inertial frames**

   (a) Using the chain rule, rewrite the $D = 2 + 1$ Minkowski line element

   $$\mathrm{d}s_M^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$$

   in polar coordinates:

   $$x = r\cos\theta, y = r\sin\theta, \tilde{t} = t.$$

   (b) Rewrite $\mathrm{d}s_M^2$ in a rotating frame, where the new coordinates are

   $$\tilde{x} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} x \equiv Rx, \quad \tilde{t} = t.$$

   (c) Redo the previous part in polar coordinates; that is, let the new coordinates be $(\tilde{t}, r, \theta)$ with the relations from part 4a but with $\theta = \omega t + \theta_0$.

   (d) Consider the action for a relativistic massive particle in ($D = 2 + 1$) Minkowski space

   $$S = mc \int \mathrm{d}\tau$$

   where τ is the proper time along the worldline, $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$. Using the action principle, derive the centripetal force experienced by a particle using uniformly rotating coordinates, $\theta = \omega t$.

5. **Is it flat?**

   Show that the two dimensional space whose metric is

   $$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2\mathrm{d}u^2$$

   (it is called 'Rindler space') is just two-dimensional Minkowski space

   $$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$$

   in disguise. Do this by finding the appropriate change of coordinates $x(u, v), t(u, v)$.