University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 5 – Solutions

Posted October 23, 2013 **Due Monday, November 4, 2013**

1. **A constant vector field.** Consider the vector field $W \equiv \partial_x$ in the flat plane with metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. $W = W_r\partial_r + W_\varphi\partial_\varphi$ and compute their partial derivatives. Then compute the (metric-compatible) covariant derivative in polar coordinates.

   The point of this problem is to convince ourselves that when we say 'a constant vector field' what we mean is 'a covariantly constant vector field', because the latter statement is meaningful independent of our choice of coordinates. The polar coordinates are $x = r\cos\varphi, y = r\sin\varphi$. Away from $r = 0$ where the polar coordinates r, φ break down, the jacobian matrix J is given by

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\varphi \end{pmatrix} \equiv J \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\varphi \end{pmatrix} .$$

   Writing this equation as $\mathrm{d}x^i = J^i_a \mathrm{d}\tilde{x}^a$, the coordinate vector fields are related by the inverse-transpose matrix:

$$\frac{\partial}{\partial x^i} = \left(J^{-1}\right)^a_i \frac{\partial}{\partial \tilde{x}^a}$$

   which is

$$\left(J^{-1}\right)^a_i = \begin{pmatrix} \cos\varphi & -\frac{\sin\varphi}{r} \\ \sin\varphi & \frac{\cos\varphi}{r} \end{pmatrix}^a_i \qquad \left(J^i_a \left(J^{-1}\right)^b_i = \delta^b_a\right)$$

   – notice that i is the row index and a is the column index – you can check this by dimensional analysis, since r and φ have different dimensions) so

$$W = \partial_x = \left(J^{-1}\right)^r_x \frac{\partial}{\partial r} + \left(J^{-1}\right)^\varphi_x \frac{\partial}{\partial \varphi} = \cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \theta}.$$

   So the partial derivatives of the components in polar coords are certainly not constant:

$$\partial_\varphi W^r = -\sin\varphi, \quad \partial_r W^r = 0, \quad \partial_\varphi W^\varphi = -\frac{\cos\varphi}{r}, \quad \partial_r W^\varphi = +\frac{\sin\varphi}{r^2}.$$

   The covariant derivative in the (flat!) plane in cartesian coordinates is just the same as the partial derivative so $\nabla_i W^j = \partial_i W^j = 0$. By construction, this must also be true in polar coordinates. More explicitly the metric in polar coordinates is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$$

which means that the nonzero Christoffel symbols are

$$\Gamma^r_{\varphi\varphi} = -r, \Gamma^\varphi_{\varphi r} = \frac{1}{r} = \Gamma^\varphi_{r\varphi} \ .$$

This means that

$$\nabla_a W^b = \begin{pmatrix} \partial_r W^r + 0 & \partial_\varphi W^r + \Gamma^r_{\varphi\varphi} W^\varphi \\ \partial_r W^\varphi + \Gamma^\varphi_{r\varphi} W^\varphi & \partial_\varphi W^\varphi + \Gamma^\varphi_{\varphi r} W^r \end{pmatrix}^b_a = \begin{pmatrix} 0 & -\sin\varphi + (-r)\frac{-\sin\varphi}{r} \\ \frac{\sin\varphi}{r^2} + \frac{1}{r}\frac{-\sin\varphi}{r} & (-\frac{\cos\varphi}{r}) + \frac{1}{r}\cos\varphi \end{pmatrix}^b_a = 0 \ \checkmark$$

(the lower index is the row index).

2. **Vector fields on the 2-sphere.** [from Ooguri] A two-dimensional sphere S^2 of unit radius can be embedded in the three-dimensional euclidean space $\mathbb{R}^3$ by the equation

$$x^2 + y^2 + z^2 = 1 \ .$$

For coordinates on the sphere we can use (θ, φ) defined by

$$x = \sin\theta\cos\varphi, \ y = \sin\theta\sin\varphi, \ z = \cos\theta,$$

except at the north and south poles $\theta = 0, \pi$ where the value of φ is ambiguous.

An infinitesimal rotation of $\mathbb{R}^3$ around its origin induces a tangent vector field on S^2, which is said to *generate* the rotation[1]. Show that there are three linearly-independent vector fields[2] of this type and compute their commutators $[\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j]$.

A rotation about the origin is a linear map $R : \mathbb{R}^3 \to \mathbb{R}^3$ which preserves the euclidean metric $\mathrm{d}s^2 \equiv \mathrm{d}x^i \mathrm{d}x^i$; if it also preserves the orientation ($\det R = 1$), then it is continuously connected to the identity map. Since such an R is linear, it is defined by its action on a(n orthonormal) basis $\hat{i}, i = 1, 2, 3$:

$$R(\hat{i}) \equiv R_{ij}\hat{j}, \quad \hat{i} \cdot \hat{j} = \delta_{ij} = R(\hat{i}) \cdot R(\hat{j}) = R_{ik}R_{kj}.$$

which gives six conditions on the nine elements R_{ij}. This means that the space of such R (the group $SO(3)$) is 3 dimensional. From the definition above, it is a closed subset of $\mathbb{R}^9$ and hence a smooth manifold. A group which is a smooth manifold is a *Lie group*. A Lie group G is generated by a Lie algebra $\mathfrak{g}$ – this is just the tangent space to G at the identity element. It has a natural product, which is the Lie bracket – the commutator of tangent vector fields. The tangent vector fields on S^2 induced by such infinitesimal rotations are therefore a realization of this Lie algebra ($so(3)$), and hence there can only be three linearly independent such vector fields. The three independent

[1] More precisely, consider the result of acting with a rotation by an infinitesimal angle θ on an arbitrary smooth function:

$$f(x) \mapsto f(Rx) = f(x + \theta Ax) = f(x) + (\theta Ax)^i \partial_i f(x) + \ldots$$

– the vector field $(Ax)^i \partial_i$ generates the rotation.

[2] A vector field v on M is linearly dependent on some others $\{v_\alpha\}$ if there exist *constants* a^α s.t. $v = a^\alpha v_\alpha$ everywhere in M.

infinitesimal rotations, which I'll denote $\mathbf{r}_i, i = 1,2,3$ can be represented on functions by vector fields v:

$$f(R(x)) = f(x) + \theta v(f)|_x + \mathcal{O}(\theta^2)$$

as follows, beginning with the rotation about the z axis:

$$(R_z)_{ij} \equiv \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} = \mathbb{1}_{ij} + \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} + \mathcal{O}(\theta^2)$$

It maps the coordinate vector x_i to

$$R_z(x)_i = (\boldsymbol{\sigma}_z)_{ij}\, x^j = \begin{pmatrix} x + \theta y \\ y - \theta x \\ z \end{pmatrix}_i + \mathcal{O}(\theta^2)$$

$$\implies f(\boldsymbol{\sigma}(x)) = f(x + \theta y, y - \theta x, z) + \mathcal{O}(\theta^2) = f(x) + \theta\,(y\partial_x - x\partial_y)\, f(x) + \mathcal{O}(\theta^2)$$
$$\implies \boldsymbol{\sigma}_z = y\partial_x - x\partial_y.$$

Cyclically permuting x, y, z, we have

$$\boldsymbol{\sigma}_x = z\partial_y - y\partial_z, \boldsymbol{\sigma}_y = x\partial_z - z\partial_x \ .$$

Writing these in terms of θ, φ, we can make clear that these restrict to vector fields tangent to $S^2 \subset \mathbb{R}^3$. This amounts to changing to polar coords and showing that these vector fields have no components along ∂_r. The Jacobian matrix (evaluated at $r = 1$) is

$$\begin{pmatrix} \partial_x\theta & \partial_x\varphi & \partial_x r \\ \partial_y\theta & \partial_y\varphi & \partial_y r \\ \partial_z\theta & \partial_z\varphi & \partial_z r \end{pmatrix} = \begin{pmatrix} \frac{zx}{\rho} & \frac{-y}{\rho^2} & \frac{x}{r} \\ \frac{zy}{\rho} & \frac{x}{\rho^2} & \frac{y}{r} \\ -\rho & 0 & \frac{z}{r} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -\frac{\sin\varphi}{\sin\theta} & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \frac{\cos\varphi}{\sin\theta} & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

(where $\rho \equiv \sqrt{x^2 + y^2}$). And so

$$\partial_x = \partial_x\theta\partial_\theta + \partial_x\varphi\partial_\varphi + \partial_x r\partial_r = \cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\partial_\varphi + \sin\theta\cos\varphi\partial_r$$

$$\partial_y = \cos\theta\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\partial_\varphi + \sin\theta\sin\varphi\partial_r$$
$$\partial_z = -\sin\theta\partial_\theta + \cos\theta\partial_r$$

And so

$$\boldsymbol{\sigma}_z = x\partial_y - y\partial_x = x\left(\frac{zy}{\rho}\right)\partial_\theta + x\frac{x}{\rho^2}\partial_\varphi + xy\partial_r - y\left(\frac{zx}{\rho}\partial_\theta - \frac{y}{\rho^2}\partial_\varphi\right) - yx\partial_r = \partial_\varphi \ .$$

$$\begin{aligned} \boldsymbol{\sigma}_x &= y\partial_z - z\partial_y = y\,(-\sin\theta\partial_\theta) - z\left(\cos\theta\sin\varphi\partial_\varphi + \frac{\cos\varphi}{\sin\theta}\partial_\varphi\right) \\ &= \left(-\sin^2\theta\sin\varphi - \cos^2\theta\sin\varphi\right)\partial_\theta - \cot\theta\cos\varphi\partial_\varphi = -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi. \end{aligned}$$

$$\boldsymbol{\sigma}_y = z\partial_x - x\partial_z = z\left(\cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\partial_\varphi\right) - x\left(-\sin\theta\partial_\theta\right) = \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi.$$

To check directly that these are independent, suppose that there exist constants a, b, c so that

$$a\boldsymbol{\sigma}_z + b\boldsymbol{\sigma}_y + c\boldsymbol{\sigma}_z = 0, \forall\theta,\varphi$$

and then evaluate at various points. At $\theta = \pi/2, \varphi = 0$ we learn $0 = a\partial_\varphi + b\partial_\theta \implies a = b = 0$. At $\theta = \pi/2, \varphi = \pi/2$ we learn that $0 = -c\partial_\theta$, so $c = 0$ as well.
It is easiest to compute the commutators in the cartesian coordinates, *e.g.*

$$[\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y] = [y\partial_z - z\partial_y, z\partial_x - x\partial_z] = y(\partial_z z)\partial_x + (-1)^3 x(\partial_z z)\partial_y = \boldsymbol{\sigma}_z$$

Cyclically permuting, we see that

$$[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j] = \epsilon_{ijk}\boldsymbol{\sigma}_k$$

which is the $so(3)$ algebra. If we are feeling like doing some penance, we can also compute the commutators directly on the sphere. For example

$$\begin{aligned}
[\boldsymbol{\sigma}_z, \boldsymbol{\sigma}_x] &= \partial_\varphi\left(-\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi\right) - \left(-\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi\right)\partial_\varphi \\
&= -\cos\varphi\partial_\theta + \cot\theta\sin\varphi\partial_\varphi = -\boldsymbol{\sigma}_y
\end{aligned}$$

which agrees with $[\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_z] = -[\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_z] = \boldsymbol{\sigma}_y$.

3. **E&M in curved space.** Consider EM fields $A_\mu(x)$ in a curved spacetime with a general metric $g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$.

   (a) Write an action functional $S[A_\mu, g_{\mu\nu}]$ which is general-coordinate invariant *and gauge invariant* and which reduces to the Maxwell action if we evaluate it in Minkowski spacetime $S[A_\mu, \eta_{\mu\nu}]$.

   $$S[A,g] = \int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + A_\mu j^\mu\right) + \frac{\theta}{16\pi^2}\int F\wedge F \quad .$$

   (b) Vary this action with respect to A_μ to find the equations of motion governing electrodynamics in curved space.
   The $F \wedge F$ term does not contribute to the equations of motion because it is a total derivative. The Maxwell term and the source term give

   $$\begin{aligned}
   0 &= \frac{\delta}{\delta A_\lambda(y)}S[A,g] = \int \mathrm{d}^4x\sqrt{g}\left(\frac{4}{4}\partial_\mu\delta^\lambda_\nu\delta(x-y)F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + j^\lambda\right) \\
   &= -\partial_\mu\left(\sqrt{g}F_{\rho\sigma}g^{\mu\rho}g^{\lambda\sigma}\right) + j^\lambda \quad . \qquad (1)
   \end{aligned}$$

   The Bianchi identity is unmodified, since it didn't depend on the metric in the first place. Note that the above action is gauge invariant if the current j is *covariantly* conserved, $\nabla_\mu j^\mu = 0$: the variation under a gauge transformation is

   $$\delta S = \int \mathrm{d}^4x\sqrt{g}\partial_\mu\lambda j^\mu \overset{\mathrm{IBP}}{=} -\int \mathrm{d}^4x\lambda\partial_\mu\left(\sqrt{g}j^\mu\right) = -\int \mathrm{d}^4x\sqrt{g}\lambda\nabla_\mu j^\mu.$$

(Here I have used our expression for the covariant divergence which does not involve Christoffel symbols explicitly.)
Note that if you chose to define the current as a *tensor density* (I don't know why you would do this other than if you were told to do so by someone on Wikipedia) which means that you have absorbed the $\sqrt{g}$ into the definition of j_μ, then your expressions would look accordingly different and ... worse. Note also that I have chosen to normalize my Maxwell field so that I didn't have to write factors of $4\pi c$.

4. **The badness cancels.** Using the coordinate transformation property of the Christoffel connection $\Gamma^\rho_{\mu\nu}$, verify that

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho$$

transforms as a rank-2 covariant tensor if ω is a one-form.

Recall that under $x \to \tilde{x}(x)$, $\Gamma \to \tilde{\Gamma}$ where

$$\tilde{\Gamma}^\rho_{\mu\nu} = \underbrace{\tilde{\partial}_\mu x^\sigma \partial_\kappa \tilde{x}^\rho \tilde{\partial}_\nu x^\delta \Gamma^\kappa_{\sigma\delta}}_{\text{tensor transf}} - \underbrace{\partial_\kappa \partial_\delta \tilde{x}^\rho \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\nu x^\delta}_{\text{extra badness}} \; . \tag{2}$$

This guarantees that $\nabla_\mu V^\nu$ is a tensor for any vector V. The first term of $\nabla_\mu \omega_\nu$ transforms according to

$$\partial_\mu \omega_\nu \to \tilde{\partial}_\mu \tilde{\omega}_\nu = \tilde{\partial}_\mu x^\kappa \partial_\kappa \left(\tilde{\partial}_\nu x^\rho \omega_\rho \right) = \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\kappa x^\rho \partial_\nu \omega_\rho + \tilde{\partial}_\mu x^\kappa (\partial_\kappa \tilde{\partial}_\nu x^\rho)\omega_\rho$$

So the whole object becomes

$$\begin{aligned} \tilde{\nabla}_\mu \tilde{\omega}_\nu = \tilde{\partial}_\mu \tilde{\omega}_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\tilde{\omega}_\rho &= \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\kappa x^\rho \partial_\nu \omega_\rho + \tilde{\partial}_\mu x^\kappa (\partial_\kappa \tilde{\partial}_\nu x^\rho)\omega_\rho \\ &+ \tilde{\partial}_\mu x^\sigma \partial_\kappa \tilde{x}^\rho \tilde{\partial}_\nu x^\delta \Gamma^\kappa_{\sigma\delta} \tilde{\partial}_\rho x^\xi \omega_\xi + \partial_\kappa \partial_\delta \tilde{x}^\rho \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\nu x^\delta \tilde{\partial}_\rho x^\xi \omega_\xi \\ &= \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\kappa x^\rho \partial_\nu \omega_\rho + \tilde{\partial}_\mu x^\kappa (\partial_\kappa \tilde{\partial}_\nu x^\rho)\omega_\rho \\ &+ \tilde{\partial}_\mu x^\sigma \partial_\xi \tilde{x}^\rho \Gamma^\xi_{\sigma\delta}\omega_\xi + \partial_\kappa \partial_\delta \tilde{x}^\rho \tilde{\partial}_\mu x^\kappa \tilde{\partial}_\nu x^\delta \tilde{\partial}_\rho x^\xi \omega_\xi \end{aligned} \tag{3}$$

– in the second step we just used the fact that the ρ index is contracted between tensors to erase a JJ^{-1}, *i.e.* $\partial_\kappa \tilde{x}^\rho \tilde{\partial}_\rho x^\xi = \delta^\xi_\kappa$. The badness will cancel if

$$-\tilde{\partial}_\kappa \tilde{\partial}_\nu x^\rho \omega_\rho = (\tilde{\partial}_\mu \partial_\delta \tilde{x}^\delta)\tilde{\partial}_\rho x^\kappa \tilde{\partial}_\nu x^\delta \omega_\delta \tag{4}$$

But on any function f,

$$[\tilde{\partial}_\mu, \partial_\delta] f = [\tilde{\partial}_\mu, \partial_\delta \tilde{x}^\rho \tilde{\partial}_\rho] f = (\tilde{\partial}_\mu \partial_\beta \tilde{x}^\rho)\tilde{\partial}_\rho f$$

and therefore, the RHS of (4) is

$$\begin{aligned} (\tilde{\partial}_\mu \partial_\delta \tilde{x}^\delta)\tilde{\partial}_\rho x^\kappa \tilde{\partial}_\nu x^\delta \omega_\delta &= \left((\tilde{\partial}_\mu \partial_\delta \tilde{x}^\rho)\tilde{\partial}_\rho x^\xi \right) \tilde{\partial}_\nu x^\delta \omega_\xi \\ &= \left([\tilde{\partial}_\mu, \partial_\delta] x^\xi \right) \tilde{\partial}_\nu x^\delta \omega_\xi \\ &= \left(\tilde{\partial}_\mu \left(\partial_\delta x^\xi \right) - \partial_\beta \left(\tilde{\partial}_\mu x^\xi \right) \right) \tilde{\partial}_\nu x^\delta \omega_\xi \end{aligned}$$

$$
\begin{aligned}
&= \left(\underbrace{\tilde{\partial}_\mu \left(\delta^\xi_\delta \right)}_{=0} - \partial_\delta \tilde{\partial}_\mu x^\xi \right) \tilde{\partial}_\nu x^\delta \omega_\xi \\
&= -\tilde{\partial}_\nu x^\delta \partial_\delta \tilde{\partial}_\mu x^\xi \omega_\xi \\
&= -\tilde{\partial}_\nu \tilde{\partial}_\mu x^\xi \omega_\xi \\
&= -\tilde{\partial}_\mu \tilde{\partial}_\nu x^\xi \omega_\xi \ .
\end{aligned} \tag{5}
$$

Probably there is a more elegant way to show this.