1. **Infinitesimal coordinate transformations.** Under a coordinate transformation, \(x^\mu \rightarrow \tilde{x}^\mu(x)\), the metric tensor transforms as

\[
g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{\partial}^\mu x^\rho \tilde{\partial}_\nu x^\sigma g_{\rho\sigma}(x) .
\]

Show that for an infinitesimal transformation \(\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)\), this takes the form

\[
\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = - (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu) .
\]

(This will be useful for the following problem.)

\[
\tilde{x}^\mu = x^\mu + \epsilon^\mu(x), \quad \tilde{\partial}^\mu x^\rho = \delta^\rho_{\mu} - \tilde{\partial}^\mu x^\alpha \partial_\alpha \epsilon^\rho = \delta^\rho_{\mu} - \partial^\rho \epsilon_\mu + O(\epsilon^2).
\]

In the following the +\(O(\epsilon^2)\) is understood, that is, we treat \(\epsilon^2\) as zero.

\[
\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = \tilde{\partial}^\mu x^\rho \big|_{x=\epsilon} \partial_\nu x^\sigma |_{x=\epsilon} g_{\rho\sigma}(x) - \epsilon - g_{\mu\nu}(x) \\
= (\delta^\rho_{\mu} - \partial^\rho \epsilon_\mu) \big(\delta^\sigma_{\nu} - \partial^\sigma \epsilon_\nu\big) g_{\rho\sigma}(x) - \epsilon \partial_\nu g_{\rho\sigma}(x) - g_{\mu\nu}(x) \\
= - (\partial^\rho \epsilon_\mu) g_{\rho\nu} - (\partial^\sigma \epsilon_\nu) g_{\mu\sigma} - \epsilon \partial_\mu g_{\rho\sigma} \\
= - (\partial^\rho \epsilon_\mu) g_{\rho\nu} - \epsilon^\rho \partial_\mu g_{\rho\nu} + \partial_\mu (\epsilon^\rho g_{\rho\nu}) - \epsilon^\sigma \partial_\nu g_{\rho\sigma} + \epsilon^\sigma \partial_\sigma g_{\mu\nu} \\
= - (\partial^\rho \epsilon_\mu + \partial_\rho \epsilon_\mu - \epsilon \partial_\mu g_{\rho\sigma} + \partial_\rho g_{\rho\sigma} - \partial_\rho g_{\mu\nu}) \\
= - (\partial^\rho \epsilon_\mu - \epsilon \partial_\mu g_{\rho\sigma} + \partial_\rho g_{\rho\sigma} - \partial_\rho g_{\mu\nu}) \\
= - (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu) .
\]

(1)

2. **Conservation of the improved stress tensor.** Our improved stress tensor is defined as

\[
T^{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}}
\]

where \(S\) is an action for a matter field (such as a scalar field \(\phi\) or a Maxwell field \(A_\mu\), or a particle trajectory). We would like to show that \(T^{\mu\nu}\) defined this way is covariantly conserved when evaluated on solutions of the equations of motion of the matter fields. For simplicity consider a scalar field \(\phi\). Its EoM is \(0 = \frac{\delta S}{\delta \phi}\).
(a) Show that under an infinitesimal coordinate transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$, the action changes as

$$\delta \epsilon S = - \int d^4x \left( \frac{1}{2} \sqrt{g} T^{\mu\nu} (\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu) + \frac{\delta S}{\delta \phi} \epsilon^\mu \partial_\mu \phi \right).$$

Using the chain rule,

$$\delta \epsilon S = \int d^4x \left( \frac{\delta S}{\delta g_{\mu\nu}(x)} \frac{\delta g_{\mu\nu}(x)}{\delta \epsilon} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu + \frac{\delta S}{\delta \phi(x)} \frac{\delta \phi(x)}{\delta \epsilon} - \epsilon^\mu \partial_\mu \phi \right).$$

Note the sign in the second term: a scalar field transforms as $\tilde{\phi}(\tilde{x}) = \phi(x)$, which means

$$\delta \phi(x) = \tilde{\phi}(x) - \phi(x) = \phi(x - \epsilon) - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x).$$

(b) By using the invariance of the action under coordinate transformations, show that the equation of motion for $\phi$ implies

$$\nabla_\mu T^{\mu\nu} = 0.$$
and the Ricci scalar is
\[ R = g^{ik} R_{ik} = \frac{1}{2} R^2 = R \]
So the einstein tensor is
\[ G_{ik} = R_{ik} - \frac{1}{2} R g_{ik} = \frac{1}{2} R g_{ik} - \frac{1}{2} R g_{ik} = 0. \]