University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 7

Posted November 4, 2013 **Due Wednesday, November 20, 2013**

1. **Infinitesimal coordinate transformations.** Under a coordinate transformation, $x^\mu \to \tilde{x}^\mu(x)$, the metric tensor transforms as[1]

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{\partial}_\mu x^\rho \tilde{\partial}_\nu x^\sigma g_{\rho\sigma}(x) \ .$$

Show that for an infinitesimal transformation $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, this takes the form

$$\delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\right).$$

(This will be useful for the following problem.)

2. **Conservation of the improved stress tensor.** Our improved stress tensor is defined as

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

where S is an action for a matter field (such as a scalar field ϕ or a Maxwell field A_μ, or a particle trajectory). We would like to show that $T^{\mu\nu}$ defined this way is covariantly conserved when evaluated on solutions of the equations of motion of the matter fields. For simplicity consider a scalar field ϕ. Its EoM is $0 = \frac{\delta S}{\delta \phi}$.

   (a) Show that under an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the action changes as

$$\delta_\epsilon S = -\int \mathrm{d}^4 x \left(\frac{1}{2}\sqrt{g} T^{\mu\nu} \left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\right) + \frac{\delta S}{\delta \phi} \epsilon^\mu \partial_\mu \phi \right).$$

   (b) By using the invariance of the action under coordinate transformations, show that the equation of motion for ϕ implies

$$\nabla_\mu T^{\mu\nu} = 0.$$

[1] Why is this relation true? The invariant distance between two nearby points doesn't care about what coordinates you use:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \tilde{g}_{\rho\sigma}(\tilde{x}) d\tilde{x}^\rho d\tilde{x}^\sigma .$$

(This equation is on page 70 of Zee's book, by the way.) Using the chain rule then gives exactly the relation above.

3. **Not so much GR in $D = 1 + 1$.**

Show that in two spacetime dimensions, the left hand side of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

vanishes identically. This means that the metric does not have any dynamics in $D = 1 + 1$, and the Einstein equations impose the constraint $T_{\mu\nu} = 0$ on the matter fields.