In the first two problems here we will think about solutions of Einstein’s equations with nonzero cosmological constant, positive and negative.

1. **AdS as a solution of Einstein’s equations.**

Consider the anti-de Sitter (AdS) metric (in so-called Poincaré coordinates)

\[
\text{d}s^2_{\text{AdS}} = L^2 \frac{dz^2 + \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu}{z^2}
\]

(1)

where \(\eta_{\mu\nu}\) is the Minkowski metric in \(e.g.\) \(D = 2 + 1\) dimensions. Show that this metric satisfies Einstein’s equations with negative cosmological constant \(\Lambda\). More specifically:

(a) Consider the action

\[
S[g] = \int \text{d}^4x \sqrt{g} (R - 2\Lambda)
\]

with \(\Lambda\) a (negative) constant. What are the equations of motion for the metric?

(b) Compute the Einstein tensor for the metric (1).

(c) Find the relation between \(\Lambda\) and \(L\) for which Einstein’s equation is solved.

(d) Optional bonus problem: determine the relation between \(L\) and \(\Lambda\) for \(AdS_D\) for arbitrary \(D\).

(e) Show that the AdS metric can also be written as

\[
\text{d}s^2_{\text{AdS}} = \text{d}y^2 + e^{-2y} \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu
\]

(2)

by the change of variables \(z = Le^{-y}\). (This form will be useful for comparison with the next problem.)

2. **de Sitter space.**

Consider the ansatz

\[
\text{d}s^2_{\text{FRW}} = -dt^2 + a(t)^2 dx^i dx^j \delta_{ij}
\]

A way to remember the correct sign (in red) here: one way to get \(\Lambda\) is to put a scalar field at the minimum of its potential everywhere in spacetime: \(\Lambda = V(\phi_0)\), but \(V(\phi)\) appears with a minus sign in the action.
where \(a(t)\) is a function we will determine.

Remarks: (1) this is the form of the FRW metric from a previous problem set, and (2) this metric is quite similar to the expression for the AdS metric in (2); specifically, they are related by the replacement \(t \rightarrow iy, y \rightarrow -it\) which exchanges a spacelike coordinate for a timelike coordinate (a ‘Wick rotation’), if we set set \(a = e^{\frac{y}{i}}\).

(a) Compute the Einstein tensor for this metric.

(b) Consider the Einstein’s equations with the stress tensor resulting from a positive cosmological constant, and find the resulting differential equation for the ‘scale factor’ \(a(t)\).

(c) Solve this equation to find the form of the FRW metric which results when the stress tensor is dominated by a positive cosmological constant, as it is presently in our universe. (Note that if we chose some other matter on the RHS we would find a different behavior of \(a(t)\).)

[Hint 1: consider the \(tt\) component of the Einstein equation first.
Hint 2: The second remark above is a good hint about the form of the solution.]

3. **Killing vector fields imply conservation laws.**

   (a) Show that Killing’s equation
   \[ \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \]  
   (3)
   is equivalent to \(\mathcal{L}_\xi g = 0\), where \(g\) is the metric tensor, and \(\mathcal{L}\) is the Lie derivative\(^2\).

   (b) If \(\xi\) is a Killing vector field (\(i.e.\) satisfies (3)) and and \(T^{\mu\nu}\) is the (covariantly conserved) energy momentum tensor, show that \(J^\mu \equiv T^{\mu\nu} \xi_\nu\) is a conserved current, \(J^\mu_\mu = 0\).

   (c) Given a time coordinate on your spacetime (and hence a notion of constant-time slices) use \(\xi\) to construct a quantity which is time-independent.

4. **Conformal coupling of a scalar field.** (This problem is optional.)

The stress tensor for a scalar field with action (in arbitrary number of dimensions \(n\)) is

\[ S_0[\phi, g] = -\frac{1}{2} \int d^n x \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \phi^2 \right) = \int d^4 x \sqrt{g} \mathcal{L} \]

is

\[ (T_0)_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S_0[\phi, g]}{\delta g^{\mu\nu}(x)} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \mathcal{L} g_{\mu\nu} \]

(a) Compute the trace, \((T_0)^\mu_\mu\).\(^3\)

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\(^2\)Reminder: The Lie derivative is a derivation, and therefore for any two vector fields, \(v_1, v_2\),

\[ \mathcal{L}_\xi (g(v_1, v_2)) = (\mathcal{L}_\xi g)(v_1, v_2) + g(L_{\xi v_1}, v_2) + g(v_1, L_{\xi v_2}). \]

\[^3\]
Recall that we associated vanishing trace of the stress tensor with scale invariance. Here we will show that it is possible to choose the constant $\gamma$ in

$$S = S_0 + \gamma \int d^n x \sqrt{g} R \phi^2$$

(where $R$ is the Ricci scalar curvature) so that this action is scale invariant.$^3$

(b) Show that for a certain value of $\gamma$ the action $S$ is invariant under the local scale transformation

$$ds^2 \rightarrow \tilde{ds}^2 = \Omega^2(x)ds^2$$

(also called a Weyl transformation or local conformal transformation (since it preserves angles)) if we also make the replacement

$$\phi(x) \rightarrow \Omega^{\frac{n-2}{2}}(x) \phi(x).$$

Find the right $\gamma$.

(c) Find the improved stress tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

and show that it is traceless.

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$^3$Hint: you may find it useful to use Appendix D of Wald. In particular, equation D.8 tells us the behavior of the Ricci tensor under a position-dependent rescaling of the metric. Specifically, if $d\tilde{s}^2 = \Omega^2(x)ds^2$ then

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - (n-2) \nabla_\mu \nabla_\nu \log \Omega - g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \nabla_\sigma \log \Omega + (n-2) (\nabla_\mu \log \Omega) \nabla_\nu \log \Omega - (n-2) g_{\mu\nu} g^{\rho\sigma} (\nabla_\rho \log \Omega) \nabla_\sigma \log \Omega. \tag{4}$$