University of California at San Diego – Department of Physics – Prof. John McGreevy

General Relativity (225A) Fall 2013
Assignment 8

Posted November 13, 2013 **Due Monday, December 2, 2013**

In the first two problems here we will think about solutions of Einstein's equations with nonzero cosmological constant, positive and negative.

1. **AdS as a solution of Einstein's equations.**

   Consider the anti-de Sitter (AdS) metric (in so-called Poincaré coordinates)

   $$ds^2_{AdS} = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} \tag{1}$$

   where $\eta_{\mu\nu}$ is the Minkowski metric in *e.g.* $D = 2+1$ dimensions. Show that this metric satisfies Einstein's equations with negative cosmological constant Λ. More specifically:

   (a) Consider the action

   $$S[g] = \int d^4x \sqrt{g} \left(R - 2\Lambda\right)$$

   with Λ a (negative) constant[1]. What are the equations of motion for the metric?

   (b) Compute the Einstein tensor for the metric (1).

   (c) Find the relation between Λ and L for which Einstein's equation is solved.

   (d) Optional bonus problem: determine the relation between L and Λ for AdS_D for arbitrary D.

   (e) Show that the AdS metric can also be written as

   $$ds^2_{\text{AdS}} = dy^2 + e^{-\frac{2y}{L}} \eta_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

   by the change of variables $z = Le^{\frac{y}{L}}$. (This form will be useful for comparison with the next problem.)

2. **de Sitter space.**

   Consider the ansatz

   $$ds^2_{\text{FRW}} = -dt^2 + a(t)^2 dx^i dx^j \delta_{ij}$$

[1] A way to remember the correct sign (in red) here: one way to get Λ is to put a scalar field at the minimum of its potential every where in spacetime: $\Lambda = V(\phi_0)$, but $V(\phi)$ appears with a minus sign in the action.

where $a(t)$ is a function we will determine.
Remarks: (1) this is the form of the FRW metric from a previous problem set, and (2) this metric is quite similar to the expression for the AdS metric in (2); specifically, they are related by the replacement $t \to \mathbf{i}y, y \to -\mathbf{i}t$ which exchanges a spacelike coordinate for a timelike coordinate (a 'Wick rotation'), if we set set $a = e^{\frac{y}{L}}$.

(a) Compute the Einstein tensor for this metric.

(b) Consider the Einstein's equations with the stress tensor resulting from a *positive* cosmological constant, and find the resulting differential equation for the 'scale factor' $a(t)$.

(c) Solve this equation to find the form of the FRW metric which results when the stress tensor is dominated by a positive cosmological constant, as it is presently in our universe. (Note that if we chose some other matter on the RHS we would find a different behavior of $a(t)$.)

[Hint 1: consider the tt component of the Einstein equation first.
Hint 2: The second remark above is a good hint about the form of the solution.]

3. **Killing vector fields imply conservation laws.**

(a) Show that Killing's equation

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{3}$$

is equivalent to $\mathcal{L}_\xi g = 0$, where g is the metric tensor, and $\mathcal{L}$ is the Lie derivative[2].

(b) If ξ is a Killing vector field (*i.e.* satisfies (3)) and and $T^{\mu\nu}$ is the (covariantly conserved) energy momentum tensor, show that $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is a conserved current, $J^\mu_{;\mu} = 0$.

(c) Given a time coordinate on your spacetime (and hence a notion of constant-time slices) use ξ to construct a quantity which is time-independent.

4. **Conformal coupling of a scalar field.** (This problem is optional.)

The stress tensor for a scalar field with action (in arbitrary number of dimensions n) is

$$S_0[\phi, g] = -\frac{1}{2}\int \mathrm{d}^n x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \phi^2\right) = \int \mathrm{d}^4 x\sqrt{g}\mathcal{L}$$

is

$$(T_0)_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S_0[\phi, g]}{\delta g^{\mu\nu}(x)} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\mathcal{L}g_{\mu\nu}.$$

(a) Compute the trace, $(T_0)^\mu_\mu$.

[2]Reminder: The Lie derivative is a derivation, and therefore for any two vector fields, $v_{1,2}$,

$$\mathcal{L}_\xi\left(g(v_1, v_2)\right) = \left(\mathcal{L}_\xi g\right)(v_1, v_2) + g(\underbrace{\mathcal{L}_\xi v_1}_{=[\xi, v_1]}, v_2) + g(v_1, \mathcal{L}_\xi v_2).$$

Recall that we associated vanishing trace of the stress tensor with scale invariance. Here we will show that it is possible to choose the constant γ in

$$S = S_0 + \gamma \int \mathrm{d}^n x \sqrt{g} R \phi^2$$

(where R is the Ricci scalar curvature) so that this action is scale invariant.[3]

(b) Show that for a certain value of γ the action S is invariant under the local scale transformation

$$\mathrm{d}s^2 \to \mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2$$

(also called a *Weyl transformation* or *local conformal transformation* (since it preserves angles)) if we also make the replacement

$$\phi(x) \to \Omega^{\frac{2-n}{2}}(x)\phi(x) \ .$$

Find the right γ.

(c) Find the improved stress tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

and show that it is traceless.

[3] Hint: you may find it useful to use Appendix D of Wald. In particular, equation D.8 tells us the behavior of the Ricci tensor under a position-dependent rescaling of the metric. Specifically, if $\mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2$ then

$$\begin{aligned} \tilde{R}_{\mu\nu} &= R_{\mu\nu} - (n-2)\nabla_\mu \nabla_\nu \log\Omega - g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \nabla_\sigma \log\Omega \\ &+ (n-2)\left(\nabla_\mu \log\Omega\right) \nabla_\nu \log\Omega - (n-2) g_{\mu\nu} g^{\rho\sigma} \left(\nabla_\rho \log\Omega\right) \nabla_\sigma \log\Omega. \end{aligned} \tag{4}$$