

General Relativity (225A) Fall 2013 Assignment 9

Posted November 22, 2013

Due Monday, December 9, 2013

This will be the last problem set.

1. Energy loss to gravitational waves.

Calculate the rate at which a binary star system loses energy to gravitational radiation. Assume two stars of equal mass M with an instantaneous separation $2r$.

2. Coordinate horizons. [from Brandenberger]

In this problem we will come to terms with the notion of an event horizon using only special relativity.

Consider a train moving with constant acceleration a in Minkowski spacetime.

[Hint 1: a covariant definition of constant acceleration is

$$a_0^2 = a^\mu a_\mu, \quad a^\mu \equiv \partial_\tau^2 x^\mu \quad (1)$$

is the proper acceleration and a_0 is a constant.

Hint 2: show that the solution of (1) is a hyperbola in the xt plane.]

- Using only special relativity, determine the frequency with which light emitted on the last car of the train is received by a detector in the front car.
- Introduce new coordinates ξ and η in which each point on the train is at constant position ξ and for which η is (proportional to) the proper time of the train. Show that the coordinate transformation is

$$t = a^{-1} e^{a\xi} \sinh(a\eta), \quad x = a^{-1} e^{a\xi} \cosh(a\eta)$$

- What is the metric in these new coordinates? Where have you seen this before?
- Use the Einstein equivalence principle to answer the question in part (a).
- Discuss the causal structure of Minkowski spacetime from the point of view of an observer on the train. Specifically, can the observer see all of Minkowski spacetime?

3. Sanity check.

The motion of the Earth about the Sun can be described as the motion of a test particle in the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2G_N M}{r} \right) dt^2 + \frac{1}{1 - \frac{2G_N M}{r}} dr^2 + r^2 ds_{S^2}^2$$

with M the mass of the Sun. Verify that the Newtonian limit of this problem gives the well-known $1/r^2$ force between the Earth and Sun. Calculate to first order in the small quantities the correction terms to the Newtonian force due to GR.

4. Falling into a black hole

A massive particle initially at rest at radius $r > 2GM$ starts falling radially into the Schwarzschild metric. Compute the proper times it takes for the particle to reach the Schwarzschild radius at $r = 2GM$ and the singularity at $r = 0$. Compute also the Schwarzschild coordinate time t to reach the horizon.

[Don't forget to take advantage of conservation laws.]

Imprecisely-formulated bonus problem: Suppose the test particle has some finite extent, ϵ . Compute and describe the tidal forces on the particle as a function of its radial position r .

The following problems are optional.

5. Killing vector fields form a closed algebra.

Show that, if ξ and η are Killing vector fields, then so is their Lie bracket $[\xi, \eta]$.

6. Exact gravitational wave solutions.

Show that with the ansatz

$$ds^2 = -dudv + \sum_{i=1,2} dy^i dy^i + F(u, y^i) du^2 \tag{2}$$

the vacuum Einstein equations are solved if

$$R_{\mu\nu} = 0 \quad \Leftrightarrow \quad \sum_i \frac{\partial^2}{\partial y^i \partial y^i} F(u, y^1, y^2) = 0. \tag{3}$$

Further, show that (4) is solved by

$$F(u, y^i) = \sum_{i,j=1,2} h_{ij}(u) y^i y^j \quad \text{if} \quad \sum_i h_{ii}(u) = 0 .$$

7. Boundary of anti-de Sitter spacetime.

Recall the metric of anti-de Sitter space (AdS) in Poincaré coordinates.

$$ds_{AdS}^2 = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} \quad (4)$$

Here the coordinate z is positive. Show that a lightray sent from a point with $z > 0$ with $\dot{z} < 0$ reaches $z = 0$ in a finite coordinate time δt .

[In this sense, $z = 0$ is a (timelike) *boundary* of the AdS spacetime, and we must specify boundary conditions there for the time evolution to be well-defined.]

8. Surface gravity and Hawking temperature.

Compute the periodicity of y for which this metric is regular at $z = z_m$ (*i.e.* has no conical deficit):

$$ds^2 = \Omega(z) \left(f(z) dy^2 + \frac{dz^2}{f(z)} + ds_{\text{other}}^2 \right)$$

where $f(z)$ is a function with a first-order zero at $z = z_m$ (*i.e.* $f(z_m) = 0$ and $\partial_z f(z_m) \neq 0$) and $\Omega(z)$ is regular and non-vanishing at $z = z_m$.

If we think of y as imaginary time, this periodicity determines the temperature of the black hole, since equilibrium at finite temperature means periodic Euclidean time.

a) Specialize your answer to the case of the euclidean Schwarzschild black hole in flat space, for which

$$f(z) = 1 - \frac{2GM}{z}, \quad \Omega = 1, \quad ds_{\text{other}}^2 = z^2 d\vec{x}^2.$$

b) Specialize your answer to the case of the euclidean AdS black hole (with planar horizon), for which

$$f(z) \equiv 1 - \frac{z^4}{z_m^4}, \quad \Omega = \frac{1}{z^2}, \quad ds_{\text{other}}^2 = d\vec{x}^2.$$

c) Show that you get the same answer by computing the ‘surface gravity’ κ of the horizon (the locus $z = z_m$), which can be defined by

$$\kappa^2 \equiv \frac{1}{2} \nabla_a \xi^b \nabla_c \xi^d g_{bd} g^{ac} \Big|_{z=z_m}$$

where ξ^a is the tangent vector to the shrinking circle, $\xi = \partial_y$.

9. Derive the energy-momentum conservation equations for an ideal gas in an FRW universe. Give a physical interpretation.
10. **Extremal Reissner-Nordstrom black hole.**

a) Consider Einstein-Maxwell theory in four dimensions, with action

$$S_{EM} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} \left(\mathcal{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ implies that

$$\mathcal{R}_{\mu\nu} = aG_N \left(2F_{\mu} \cdot F_{\nu} - \frac{1}{2} g_{\mu\nu} F^2 \right)$$

for some constant a .

b) Consider the ansatz

$$ds^2 = H^{-2}(\rho) (-dt^2) + H^2(\rho) (d\rho^2 + \rho^2 d\Omega_2^2),$$

$$F = bdt \wedge d(H(\rho)^{-1})$$

where b is some constant. Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$ and Maxwell's equation $0 = \frac{\delta S_{EM}}{\delta A_{\mu}}$ are solved by the ansatz if H is a harmonic function on the \mathbb{R}^3 whose metric is

$$\gamma_{ab} dx^a dx^b := d\rho^2 + \rho^2 d\Omega_2^2.$$

Recall that H is harmonic iff $0 = \square H = \frac{1}{\sqrt{\gamma}} \partial_a (\sqrt{\gamma} \gamma^{ab} \partial_b H)$.

c) Find the form of the harmonic function which gives a spherically symmetric solution; fix the two integration constants by demanding that i) the spacetime is asymptotically flat and ii) the black hole has charge Q , meaning $\int_{S^2} \text{at fixed } \rho \star F = Q$.

d) Take the near-horizon limit. Show that the geometry is $AdS_2 \times S^2$. Determine the relationship between the size of the throat and the charge of the hole.

e) If you're feeling brave, add some magnetic charge to the black hole. You will need to change the form of the gauge field to

$$F = bdt \wedge dH(\rho) + G(\rho)\Omega_2$$

where Ω_2 is the area 2-form on the sphere, and G is some function.