University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015 Assignment 2

Due 12:30pm Wednesday, October 14, 2015

1. **Paulis**

   (a) Show (from the matrix representations) that
   $$\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j = \mathbf{i}\epsilon_{ijk}\boldsymbol{\sigma}^k + \delta_{ij}\mathbb{1} \tag{1}$$
   where ϵ_{ijk} is the completely antisymmetric tensor in three indices, with
   $$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1, \epsilon_{231} = \epsilon_{132} = \epsilon_{321} = -1$$
   and $\epsilon_{ijk} = 0$ if any two indices are equal. (On the right hand side of (1), we are using the Einstein summation convention.)

   (b) Use (1) to show that
   $$(\check{n} \cdot \vec{\boldsymbol{\sigma}})^2 = \mathbb{1}$$
   if $\check{n}$ is a unit vector.

   (c) Show by series expansion that
   $$e^{\mathbf{i}\check{n}\cdot\vec{\boldsymbol{\sigma}}\theta} = \cos\theta\mathbb{1} + \mathbf{i}\sin\theta\check{n}\cdot\boldsymbol{\sigma}.$$

2. **Projector onto spin-up along $\check{n}$.**

   Show that the projector onto the state of a qbit $|\uparrow_{\check{n}}\rangle$ with spin up along an arbitrary unit vector $\check{n}$ (defined by the relation $\check{n}\cdot\vec{\boldsymbol{\sigma}}|\uparrow_{\check{n}}\rangle = |\uparrow_{\check{n}}\rangle$)[1] can be written as
   $$|\uparrow_{\check{n}}\rangle\langle\uparrow_{\check{n}}| = \frac{1}{2}\left(1 + \check{n}\cdot\boldsymbol{\sigma}\right) .$$
   That is: check that the operator on the RHS is hermitian, squares to itself, and acts correctly on a basis.

3. **Functions of operators.**

   Given a linear operator A, define the 'superoperator' ad_A, which implements the adjoint action of A, by
   $$\mathrm{ad}_A(B) = [A, B].$$

[1] Sometimes it is called $|+_{\check{n}}\rangle \equiv |\uparrow_{\check{n}}\rangle$.

(a) Show that this operator is a derivation,

$$\mathrm{ad}_A(BC) = (\mathrm{ad}_A B)C + B(\mathrm{ad}_A C)\ . \tag{2}$$

That is: it satisfies a product rule, like a derivative.

(b) Show that (for finite-dimensional Hilbert spaces) one can "integrate by parts" under the trace:

$$\mathrm{tr}\left[B \mathrm{ad}_A C\right] = -\mathrm{tr}\left[(\mathrm{ad}_A B)\, C\right].$$

(c) We are going to derive the formula

$$e^{A+B} = e^A e^{\hat{C}_A(B)}, \tag{3}$$

where

$$\hat{C}_A = \int_0^1 ds\ e^{-s\mathrm{ad}_A} = \frac{e^{\mathrm{ad}_A} - 1}{\mathrm{ad}_A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \mathrm{ad}_A^n = 1 - \frac{1}{2}\mathrm{ad}_A + \frac{1}{6}\mathrm{ad}_A^2 + \dots\ .$$

To do this, consider the operator-valued function

$$G(s) = e^{-sA} e^{s(A+B)}.$$

Derive a first-order differential equation for $G(s)$ and solve it.

(d) For the special case where $[A, B] = c$ is a c-number show that (4) implies

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}c}. \tag{4}$$

4. **Time evolution of a two-level system.** [Le Bellac]

Consider the Hamiltonian

$$H = \hbar \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

in the basis $|0\rangle \equiv |\uparrow\rangle, |1\rangle \equiv |\downarrow\rangle$, where A, B are real numbers. Set $\hbar = 1$. You know the eigensystem of this matrix from our discussion of Pauli gymnastics. It is useful to write it in terms of Ω, θ defined by

$$\Omega = 2\sqrt{A^2 + B^2}, \quad \tan\theta = \frac{B}{A}\ .$$

(a) The state vector at time t can be written as

$$|\varphi(t)\rangle = c_+(t)|0\rangle + c_-(t)|1\rangle.$$

Find a system of ODEs for the components $c_\pm(t)$ that follow from the Schrödinger equation.

(b) Decompose the initial state $|\varphi(t=0)\rangle$ in the energy eigenbasis ($H|\epsilon_\pm\rangle = \epsilon_\pm|\epsilon_\pm\rangle$):

$$|\varphi(t=0)\rangle = \sum_\pm a_\pm |\epsilon_\pm\rangle \ .$$

Express $c_+(t)$ in terms of the coefficients $a_\pm$.

(c) Suppose $c_+(0) = 0$. Find $a_\pm$ (up to a phase) in terms of A, B (or rather, Ω and θ). Find the probability of finding the system in the state $|\uparrow\rangle$ at time t.

5. **The variational method.** [Le Bellac]

(a) Let $|\varphi\rangle$ be a vector (not necessarily normalized) in the Hilbert space of states, and let $\mathbf{H}$ be a Hamiltonian of interest. The expectation value $\langle \mathbf{H} \rangle_\varphi$ is

$$\langle \mathbf{H} \rangle_\varphi = \frac{\langle \varphi | \mathbf{H} | \varphi \rangle}{\langle \varphi | \varphi \rangle}.$$

Show that if the minimum and maximum of this quantity on the space of states are obtained for $|\varphi\rangle = |\varphi_m\rangle$ and $|\varphi\rangle = |\varphi_M\rangle$ respectively, then $|\varphi_{m,M}\rangle$ are eigenvectors of $\mathbf{H}$ with the smallest and largest eigenvalues in the spectrum, E_m, E_M.

(b) Suppose the vector $|\varphi\rangle$ depends on a parameter α: $|\varphi\rangle = |\varphi(\alpha)\rangle$. Show that if

$$\partial_\alpha \langle \mathbf{H} \rangle_{\phi(\alpha)} |_{\alpha=\alpha_0} = 0$$

then $E_m \leq \langle \mathbf{H} \rangle_{\varphi(\alpha_0)}$ if α_0 is a minimum of $\langle \mathbf{H} \rangle_{\varphi(\alpha)}$. Similarly, show that $E_M \geq \langle \mathbf{H} \rangle_{\varphi(\alpha_0)}$ if α_0 is a maximum of $\langle \mathbf{H} \rangle_{\varphi(\alpha)}$.

This fact forms the basis of an approximation method called the variational method. If you have a good guess for the form of the groundstate, you can find the best approximation within that family of states, and it produces a bound on the correct groundstate energy.

(c) Consider the special case of $\dim \mathcal{H} = 2$, with Hamiltonian

$$\mathbf{H} = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix}$$

with a, b, c real. Parametrizing the variational state as

$$|\varphi(\alpha)\rangle = \begin{pmatrix} \cos\alpha/2 \\ \sin\alpha/2 \end{pmatrix},$$

find the values of $\alpha = \alpha_0$ which extremize the expectation value of the Hamiltonian. Show that this reproduces the eigenstates:

$$|\chi_+\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}, \quad |\chi_-\rangle = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}.$$

where $\tan\theta = b/c$.

6. **The Feynman-Hellmann theorem.** Suppose the Hamiltonian depends on a parameter s, $\mathbf{H} = \mathbf{H}(s)$. Suppose $E(s)$ is a non-degenerate eigenvalue with normalized eigenvector $|\phi(s)\rangle$. Show that

$$\partial_s E(s) = \langle \phi(s) | \partial_s \mathbf{H}(s) | \phi(s) \rangle.$$