University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015
Assignment 3

Due 12:30pm Wednesday, October 21, 2015

1. **Neutrino oscillations** [This problem has some overlap with a problem from the previous homework, but let's celebrate this year's Nobel Prize in Physics.]

   Neutrinos are subatomic particles which come in several types; different types interact differently with matter. Consider a neutrino which can be an electron-neutrino ν_e or a muon-neutrino ν_μ. We can describe its wave function as $|\psi\rangle = a|\nu_e\rangle + b|\nu_\mu\rangle$, where $|\nu_e\rangle, |\nu_\mu\rangle$ are orthonormal. A neutrino in the state $|\nu_e\rangle$ or $|\nu_\mu\rangle$ is detected as an electron-neutrino or muon-neutrino, respectively, with certainty. The Hamiltonian is

   $$\mathbf{H} = p + \frac{\mathbf{M}^2}{2p}$$

   where p is the (magnitude of the) momentum, and $\mathbf{M}^2$ is a 2×2 matrix with eigenvectors

   $$|\nu_1\rangle = \cos\theta|\nu_e\rangle + \sin\theta|\nu_\mu\rangle$$

   $$|\nu_2\rangle = -\sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle$$

   with respective eigenvalues m_1^2 and m_2^2. (We work in units with $c = 1$ for this problem.)

   (a) Evaluate

   $$c_i^e \equiv \langle\nu_i|\nu_e\rangle \quad \text{and} \quad c_i^\mu \equiv \langle\nu_i|\nu_\mu\rangle, \quad i = 1, 2$$

   as functions of θ.

   (b) At $t = 0$ a neutrino with momentum p is in the state $|\nu_e\rangle$. If we were to measure the observable $\mathbf{M}^2$, what are the possible outcomes and with what probabilities will they occur?

   (c) At $t = 0$ a neutrino with momentum p is in the state $|\nu_e\rangle$. At time t it reacts with matter inside of a detector. What is the probability that it is detected as a muon-neutrino?

   [Hint: I found it useful to introduce the notation $\alpha_i \equiv t\left(p + \frac{m_i^2}{2p}\right)$.]

(d) [bonus] Restore the factors of c.

2. **Density operator is non-negative.**

   Show that even if the states $|\alpha_i\rangle$ are not orthogonal, the density operator $\boldsymbol{\rho} = \sum_i p_i |\alpha_i\rangle\langle\alpha_i|$ is *non-negative*, in the sense that

   $$\langle\psi|\boldsymbol{\rho}|\psi\rangle \geq 0 \quad \forall\psi.$$

   Check that this condition implies that all eigenvalues of $\boldsymbol{\rho}$ are non-negative.

3. **Density operators for two-state systems.**

   For the case of a single qbit, the most general density matrix is a hermitian 2×2 matrix, so can be written

   $$\boldsymbol{\rho} = n_0 \mathbb{1} + \frac{1}{2} n_i \boldsymbol{\sigma}^i \ .$$

   Find the restrictions on the n_a from the properties of the density matrix. What is the geometry of the space of allowed $\vec{n}$? Which ones are the pure states?

4. **A density operator is a probability distribution on quantum states.** [from A. Manohar]

   An electron gun produces electrons randomly polarized with spins up or down along one of the three possible radomly selected orthogonal axes $1, 2, 3$ (i.e. x, y, z), with probabilities

   $$p_{i,\uparrow} = \frac{1}{2} d_i + \frac{1}{2} \delta_i \qquad p_{i,\downarrow} = \frac{1}{2} d_i - \frac{1}{2} \delta_i \qquad i = 1, 2, 3 \tag{1}$$

   Probabilites must be non-negative, so $d_i \geq 0$ and $|\delta_i| \leq d_i$.

   (a) Write down the resultant density matrix ρ describing our knowledge of the electron spin, in the basis $|\uparrow\rangle$, $|\downarrow\rangle$ with respect to the z axis.

   (b) Any 2×2 matrix ρ can be written as

   $$\rho = a\, \mathbb{1} + \vec{b} \cdot \vec{\boldsymbol{\sigma}} \tag{2}$$

   in terms of the unit matrix and 3 Pauli matrices. Determine a and $\vec{b}$ for ρ from part (a).

   (c) A second electron gun produces electrons with spins up or down along a single axis in the direction $\check{n}$ with probabilities $(1 \pm \Delta)/2$. Find $\check{n}$ and Δ so that the electron ensemble produced by the second gun is the same as that produced by the first gun.

5. **Purity test.** [from Chuang and Nielsen]

   Show that for any density matrix $\boldsymbol{\rho}$:

   (a) $\mathrm{tr}\boldsymbol{\rho}^2 \leq 1$

   (b) the inequality is saturated only if $\boldsymbol{\rho}$ is a pure state.

   [Hint: don't forget that the trace operation is basis-independent.]

6. **Polarization.** [from Boccio]

   In the expression for the general density matrix of a qbit

$$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{P}\cdot\vec{\boldsymbol{\sigma}}\right)$$

   the vector $\vec{P}$ is called the *polarization*.

   (a) To justify this name, show that it gives the expectation for the spin operator in the state $\boldsymbol{\rho}$:

$$\vec{P} = \langle\vec{\boldsymbol{\sigma}}\rangle_{\boldsymbol{\rho}}.$$

   (b) Subject the qbit to a external magnetic field, which couples by

$$\mathbf{H} = -\vec{\boldsymbol{\mu}}\cdot\vec{B} = -\frac{\gamma}{2}\boldsymbol{\sigma}\cdot\vec{B}.$$

   (γ is called the *gyromagnetic ratio.*) Assuming the qbit is isolated, so that its time evolution is unitary, what is the time evolution of the polarization, $\partial_t\vec{P}$?

7. **von Neumann entropy.**

   Consider again the general density matrix for a qbit,

$$\boldsymbol{\rho}_{\vec{n}} = \frac{1}{2}\left(\mathbb{1} + \vec{n}\cdot\boldsymbol{\sigma}\right).$$

   Recall that $\vec{n}$, $\vec{n}\cdot\vec{n} \leq 1$, specifies a point on the *Bloch ball.*

   Compute the von Neumann entropy $S(\boldsymbol{\rho}_{\vec{n}})$ of this operator; it is defined in general as

$$S(\boldsymbol{\rho}) \equiv -\mathrm{tr}\boldsymbol{\rho}\log\boldsymbol{\rho}.$$

   Express the answer as a function of a single variable. Interpret this variable in terms of the geometry of the space of states.

   [Hint: use the fact that the von Neumann entropy is independent of the choice of basis.]