University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015
Assignment 4

Due 12:30pm Wednesday, October 28, 2015

1. **Minimum uncertainty state.** [from Commins] The Heisenberg uncertainty relation for a coordinate and its conjugate momentum

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

is derived from the commutation relation $[\mathbf{x}, \mathbf{p}] = \mathbf{i}\hbar$ with the aid of the Cauchy-Schwarz inequality. From consideration of the conditions that must be satisfied for that inequality to become an equality, show that the minimum uncertainty wavefunction must be a Gaussian.

2. **Time-energy uncertainty.** Consider an observable $\mathbf{A}$ which has no explicit dependence on time. Define the variance of $\mathbf{A}$ in a state ψ to be

$$\Delta A \equiv \sqrt{\langle (\mathbf{A} - \langle \mathbf{A} \rangle_\psi)^2 \rangle_\psi}$$

as usual, with $\langle \mathbf{A} \rangle_\psi \equiv \langle \psi | \mathbf{A} | \psi \rangle$. Let

$$\Delta T \equiv \frac{\Delta A}{|\partial_t \langle \mathbf{A} \rangle_\psi|}.$$

This is a measure of the time required for $\langle \mathbf{A} \rangle$ to change significantly, *i.e.* by an amount comparable to its variance ΔA. Show that

$$\Delta E \Delta T \geq \frac{\hbar}{2},$$

where ΔE is the variance of $\mathbf{H}$, the Hamiltonian.

3. **Wigner distribution function.** [from Commins] Let $\psi(x,t) = \langle x | \psi(t) \rangle$ be a position-space wavefunction for a particle in one dimension, and let $\phi(p,t) = \langle p | \psi(t) \rangle$ be its momentum-space counterpart. Define the associated *Wigner* (or *phase-space*) *distribution function* by

$$W(x,p,t) \equiv \frac{1}{\pi} \int dy e^{2\mathbf{i}py} \psi^\star(x+y,t)\psi(x-y,t).$$

The arguments of W, x and p are c-numbers. (The dependence on time will not play an important role in our discussion.)

(a) Show that the probability density in position space is

$$\rho(x,t) \equiv |\psi(x,t)|^2 = \int dp W(x,p,t).$$

(b) Evaluate the Wigner function for a gaussian wavefunction, $\psi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$.

(c) Show that the probability density in momentum space is

$$|\phi(p,t)|^2 = \int dx W(x,p,t).$$

(d) Let $W_1(x,p,t), W_2(x,p,t)$ be the Wigner functions associated with two states $|\psi_1(t)\rangle, |\psi_2(t)\rangle$. Show that their overlap-squared is

$$|\langle\psi_1|\psi_2\rangle|^2 = 2\pi \int dxdp\ W_1^\star(x,p,t)W_2(x,p,t).$$

4. [optional] Can a general operator acting on the Hilbert space of a particle on a line be written formally in terms of the position and wavenumber ($\mathbf{k} = -\mathbf{i}\boldsymbol{\partial}$) operators as

$$\mathbf{A} = \sum_{nm} a_{nm}\mathbf{k}^n\mathbf{x}^m\ ?$$

If so, find the condition on a_{nm} for $\mathbf{A}$ to be hermitian.

5. [optional] Is the uncertainty principle still true for mixed states? If so, prove it. I suggest the form

$$\Delta_\rho A \Delta_\rho B \overset{?}{\geq} \frac{1}{2}|\langle[\mathbf{A},\mathbf{B}]\rangle_\rho$$

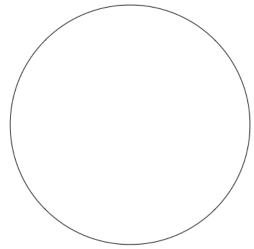
with

$$\langle\mathbf{A}\rangle_\rho \equiv \mathrm{tr}\boldsymbol{\rho}\mathbf{A}, \quad \Delta_\rho A \equiv \sqrt{(\mathbf{A}-\langle\mathbf{A}\rangle_\rho)^2}.$$

The following three problems form a triptych, on the subject of resolving the various infinities involved in the quantum mechanics of a particle on the real line. There are two such infinities: one is the fact that the real line goes on forever; this is resolved in problem 6. The other is the fact that in between any two points there are infinitely many points; this is resolved in problem 7. In problem 8 we resolve both to get a finite-dimensional Hilbert space.

6. **Particle on a circle.**

Consider a particle which lives on a circle:



That is, its coordinate x takes values in $[0, 2\pi R]$ and we identify $x \simeq x + 2\pi R$.

(a) Let's assume that the wavefunction of the particle is periodic in x:

$$\psi(x + 2\pi R) = \psi(x) \ .$$

What set of values can its momentum (that is, eigenvalues of the operator $\mathbf{p} = -i\hbar\partial_x$) take?

(b) Recall that the overall phase of the state vector is not physical data. This suggests the possibility that the wavefunction might not be periodic, but instead might acquire a phase when we go around the circle:

$$\psi(x + 2\pi R) = e^{i\varphi}\psi(x)$$

for some fixed φ. In this case what values does the momentum take?

7. **Particle on a lattice.**

Now consider a particle which lives on a lattice: its position can take only the discrete values $x = na, n \in \mathbb{Z}$ where a is some unit of length and n is an integer. We'll call the corresponding position eigenstates $|n\rangle$. The Hilbert space is still infinite-dimensional, but at least we have in our hands a countably infinite basis.

In this problem we will determine: what is the spectrum of the momentum operator $\mathbf{p}$ in this system?

(a) Consider the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} e^{in\theta} |n\rangle.$$

Show that $|\theta\rangle$ is an eigenstate of the *translation operator* $\hat{T}$, defined by

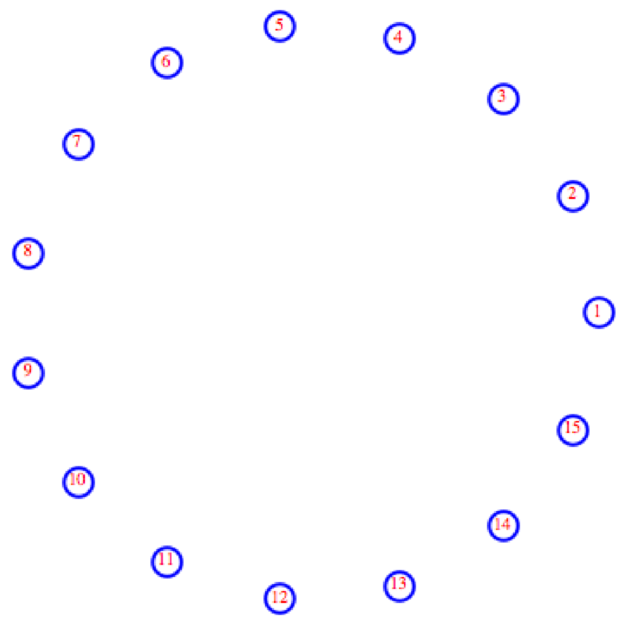
$$\hat{T} = \sum_{n \in \mathbb{Z}} |n+1\rangle\langle n|.$$

Why do I want to call θ momentum?

(b) What range of values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

8. **Discrete Laplacian.**

Consider again a particle which lives on a lattice, but now we'll wrap the lattice around a circle, in the following sense. Its position can take only the discrete values $x = a, 2a, 3a, ..., Na$ (where, again, a is some unit of length and again we'll call the corresponding position eigenstates $|n\rangle$). Suppose further that the particle lives on a circle, so that the site labelled $x = (N+1)a$ is the same as the site labelled $x = a$. We can visualize this as in the figure:



In this case, the Hilbert space has finite dimension N.

Consider the following $N \times N$ matrix representation of a Hamiltonian operator (a is a constant):

$$H = \frac{1}{a^2}\left(\left.\underbrace{\begin{matrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{matrix}}_{N}\right\} N \right)$$

(a) Convince yourself that this is equivalent to the following: Acting on an N-dimensional Hilbert space with orthonormal basis $\{|n\rangle, n = 1, \dots, N\}$, $\hat{H}$ acts by

$$a^2 \hat{H}|n\rangle = 2|n\rangle - |n+1\rangle - |n-1\rangle, \quad \text{with } |N+1\rangle \simeq |1\rangle$$

that is, we consider the arguments of the ket to be integers modulo N.

(b) Show that $\hat{H}$ and $\hat{T}$ (where $\hat{T}$ is the 'shift operator' defined by $\hat{T} : |n\rangle \mapsto |n+1\rangle$) can be simultaneously diagonalized.

Consider again the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle.$$

(c) Show that $|\theta\rangle$ is an eigenstate of $\hat{T}$, for values of θ that are consistent with the periodicity $n \simeq n + N$.

(d) What values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

(e) Find the matrix elements of the unitary operator $\mathbf{U}$ which relates position eigenstates $|n\rangle$ to momentum eigenstates $|\theta\rangle$: $U_{\theta n} \equiv \langle n|\theta\rangle$.

(f) Find the spectrum of $\hat{H}$.

Draw a picture of $\epsilon(\theta)$: plot the energy eigenvalues versus the 'momentum' θ.

(g) Show that the matrix above is an approximation to (minus) the 1-dimensional Laplacian $-\partial_x^2$. That is, show (using Taylor's theorem) that

$$a^2 \partial_x^2 f(x) = -2f(x) + (f(x+a) + f(x-a)) + \mathcal{O}(a)$$

(where "$\mathcal{O}(a)$" denotes terms proportional to the small quantity a).

(h) In the expression for the Hamiltonian, to restore units, I should have written:

$$\hat{H}|n\rangle = \frac{\hbar^2}{2m} \frac{1}{a^2} \left(2|n\rangle - |n+1\rangle - |n-1\rangle\right), \qquad \text{with } |N+1\rangle \simeq |1\rangle$$

where a is the distance between the sites, and m is the mass. Consider the limit where $a \to 0, N \to \infty$ and look at the lowest-energy states (near $p = 0$); show that we get the spectrum of a free particle on the line, $\epsilon = \frac{p^2}{2m}$.