

Quantum Mechanics (Physics 212A) Fall 2015

Assignment 6

Due 5pm Thursday, November 12, 2015

1. Coherent states.

Consider a quantum harmonic oscillator with frequency ω . The creation and annihilation operators \mathbf{a}^\dagger and \mathbf{a} satisfy the algebra

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1$$

and the vacuum state $|0\rangle$ satisfies $\mathbf{a}|0\rangle = 0$. Coherent states are eigenstates of the annihilation operator:

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

(a) Show that

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \mathbf{a}^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

is an eigenstate of \mathbf{a} with eigenvalue α . (\mathbf{a} is not hermitian, so its eigenvalues need not be real.)

(b) Coherent states with different α are not orthogonal. (\mathbf{a} is not hermitian, so its eigenstates need not be orthogonal.) Show that $|\langle \alpha_1 | \alpha_2 \rangle|^2 = e^{-|\alpha_1 - \alpha_2|^2}$.

(c) Compute the expectation value of the number operator $\mathbf{n} = \mathbf{a}^\dagger \mathbf{a}$ in the coherent state $|\alpha\rangle$.

(d) Time evolution acts nicely on coherent states. The hamiltonian is $\mathbf{H} = \hbar\omega (\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2})$. Show that a coherent state evolves into a coherent state with an eigenvalue $\alpha(t)$:

$$e^{-i\mathbf{H}t} |\alpha\rangle = e^{-i\omega t/2} |\alpha(t)\rangle$$

where $\alpha(t) = e^{-i\omega t} \alpha$.

(e) Show that the coherent states can be used to resolve the identity in the form

$$\mathbb{1} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha|$$

where $d^2\alpha \equiv d\alpha_1 d\alpha_2$ in terms of the real and imaginary parts of $\alpha = \alpha_1 + i\alpha_2$. One way to do this is to relate this expression to $\mathbb{1} = \sum_{n=0}^{\infty} |n\rangle \langle n|$.

2. Oscillation of excited oscillator states.

Consider a 1d harmonic oscillator of frequency ω . Consider the initial state

$$|\psi_{n,s}(0)\rangle \equiv \mathbf{T}(s)|n\rangle$$

where $|n\rangle \equiv \frac{1}{\sqrt{n!}} (\mathbf{a}^\dagger)^n |0\rangle$ is the n th excited state and $\mathbf{T}(s) \equiv e^{-i\mathbf{P}s}$ is the displacement operator.

Describe (plot it as a function of q for some $n, t, s > 0$) the time evolution of the probability distribution: $\rho(q, t) = |\psi_{n,s}(q, t)|^2$ where $\psi_{n,s}(q, t) \equiv \langle q|e^{-i\mathbf{H}t}|\psi_{n,s}(0)\rangle$. Does it keep its shape like it does for $n = 0$?

3. Locally linear confining potential.

Determine the spectrum of the hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + V(\mathbf{x})$$

with $V(x) = a|x|$. (You can give your answer in terms of the (not analytically-determinable) locations of zeros of a special function.)

4. Particle in a uniform force field.

- Find the propagator $K(x, x'; t) \equiv \langle x'|e^{-i\mathbf{H}t}|x\rangle$ for a particle in a uniform force field, that is, $\mathbf{H} = \mathbf{p}^2 + g\mathbf{x}$.
- Use the result of the first part to find the time evolution of an initial minimal-uncertainty wavepacket.
- Show that in the accelerating frame in which there is no field, the resulting wavepacket does not accelerate, but does disperse.

The first two steps of this problem can be done using Gaussian integrals. The first step will require the BCH formula.