University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics (Physics 212A) Fall 2015
Assignment 8

Due 12:30pm Wednesday, November 25, 2015

1. **Probability current for charged particle**

   Check that probability is still conserved $\partial_t \rho + \vec{\nabla} \cdot \vec{j}_A = 0$ for a charged particle, with $\rho(x,t) = |\psi(x,t)|^2$ and

   $$\vec{j}_A \equiv \frac{\hbar}{2m\mathbf{i}} \left(\psi^\star \vec{\nabla} \psi - \psi \vec{\nabla} \psi^\star \right) - \frac{e}{mc} \psi^\star \psi \vec{A}.$$

   Check that $\vec{j}_A$ is gauge invariant if the gauge transformation acts on the wavefunction by

   $$\vec{A} \to \vec{A} + \vec{\nabla}\lambda, \quad \psi \to e^{\pm \mathbf{i} \frac{e\lambda}{\hbar c}} \psi$$

   for one choice of the sign in the exponent.

   (As a check of the sign, you can check that the Schrödinger equation maps to itself under the transformation.)

2. **Landau levels** [Shankar]

   In this problem we will consider a charged particle in a uniform magnetic field $\vec{B} = B\check{z}$. We will ignore the dimension in which the field is pointed, so the particle moves only in the two directions x, y transverse to the field. This problem is a crucial ingredient in the quantum Hall effect(s).

   Consider a particle of charge q in a vector potential

   $$\vec{A} = \frac{B}{2} \left(-y\check{x} + x\check{y} \right).$$

   (a) Show that the magnetic field is as stated above.

   (b) Show that a classical particle in this potential will move in circles at an angular frequency $\omega_0 = \frac{qB}{mc}$ where m is the mass.

   (c) Consider the Hamiltonian for the corresponding quantum problem

   $$\mathbf{H} = \frac{1}{2m} \left(\left(\mathbf{p}_x + \frac{qB}{2c} \mathbf{y} \right)^2 + \left(\mathbf{p}_y - \frac{qB}{2c} \mathbf{x} \right)^2 \right).$$

Show that

$$\mathbf{Q} \equiv \frac{1}{qB}\left(c\mathbf{p}_x + \frac{qB}{2}\mathbf{y}\right) \text{ and } \mathbf{P} \equiv \left(\mathbf{p}_y - \frac{qB}{2c}\mathbf{y}\right)$$

are canonical in the sense that $[\mathbf{Q}, \mathbf{P}] = \mathbf{i}\hbar$. Write $\mathbf{H}$ in terms of these operators and show that the allowed levels are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$.

(d) What is the multiplicity of each of these energy eigenvalues? Hint: find another canonical pair of operators that commutes with $\mathbf{H}$ and with $\mathbf{Q}, \mathbf{P}$.

(e) To understand the degeneracy better, let's write the wavefunctions for $n = 0$ (the *lowest Landau level* (LLL)) in terms of $z \equiv x + \mathbf{i}y, z^\star \equiv x - \mathbf{i}y$. Recall that the groundstate(s) of a harmonic oscillator satisfy $\mathbf{a}|0\rangle = 0$. Write this condition for the $n = 0$ states in terms of $z, z^\star$. Writing the LLL wavefunctions as

$$\psi_0(z, z^\star) = \langle x, y|n = 0\rangle = e^{-\frac{qB}{4\hbar c}zz^\star}u(z, z^\star)$$

show that the condition is solved when $u(z, z^\star)$ is any *holomorphic* function: $\partial_{z^\star}u = 0$.

(f) [optional] A useful basis of such functions is monomials $u_m = z^m$. Show that $\psi_{0,m} \equiv z^m e^{-\frac{1}{4\ell_B^2}zz^\star}$ (where $\ell_B \equiv \frac{\hbar c}{qB}$ is the *magnetic length*) is peaked for large m at a radius $r_m = \sqrt{2m}\ell_B$.

(g) [optional] Show that $\psi_{0,m}$ is an eigenstate of the angular momentum $\mathbf{L}_z = \mathbf{i}\left(\mathbf{x}\mathbf{p}_y - \mathbf{y}\mathbf{p}_x\right) = \mathbf{i}\hbar\partial_\varphi$, where $z \equiv re^{\mathbf{i}\varphi}$.

(h) [optional] If the system is a disc of radius R there is a biggest value of m that can fit. Show that the number of LLL states that can fit is

$$N = \frac{\Phi_B}{\Phi_0}$$

where $\Phi_B = \pi R^2 B$ is the flux through the sample and $\Phi_0 \equiv \frac{2\pi\hbar c}{q}$ is the flux quantum which appeared in the periodicity of the interference pattern in the Aharonov-Bohm experiment.

3. **Aharonov-Casher effect** [Commins]

Consider a neutral particle with spin-$\frac{1}{2}$ (such as a neutron), described by the Lagrangian

$$L = \frac{1}{2}mv^2 + \mu\vec{\sigma} \cdot \left(\vec{v} \times \vec{E}\right)$$

where $\vec{\sigma}$ is the spin operator and $\vec{v} \equiv \dot{\vec{x}}$.

Find the canonical momentum and the Hamiltonian.

Consider an experiment where a cylindrically-symmetric electric field $\vec{E}$ is produced by a line charge with charge-per-unit-length λ extended in the $\check{z}$ direction. Two beams of the particles described by L are sent in paths around the line charge and allowed to interfere. Show that the phase shift between the two waves arising from the line charge is

$$\delta_{\pm} = \pm \frac{4\pi\lambda\mu}{\hbar c}$$

for spin up/down in the $\check{z}$ basis.