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Quantum Mechanics C (Physics 212A) Fall 2015 Worksheet 1

Announcements

- The 212A web site is:

  http://physics.ucsd.edu/∼mcgreevy/f15/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week is Dirac notation/linear algebra bootcamp.

Problems

1. **Compatible Measurements**

   Consider the following matrices: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & 3a \\ 6 & b \end{pmatrix}$

   (a) For what values of a and b are these simultaneously diagonalizable?
   We define the *commutator* $[M, N] = MN - NM$

   (b) For these particular values of a and b what is $[X, Q]$?

   (c) Does $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ work?

2. **Projecting**

   We say P is a projector if $P^2 = P$; projecting twice is the same as once.

   A hermitian projection is one where $P^\dagger = P$ [1]

   (a) Prove that $(\mathbb{1} - P)$ is also a projector

   (b) Show that $(\mathbb{1} - P)$ and P are complementary.
   That is: $\text{range}(\mathbb{1} - P) = \text{null}P$ and $\text{null}(\mathbb{1} - P) = \text{range}P$
   This shows that every vector $|\psi\rangle$ can be written as $|\psi\rangle = P|\psi\rangle + (\mathbb{1} - P)|\psi\rangle$

   (c) Show that for every normalized $|\psi\rangle$ that $0 \leq \langle\psi|P|\psi\rangle \leq 1$

   (d) Show that a hermitian projector can never lengthen a vector.

[1] These are also called orthogonal.

3. **Represent**

Suppose the vectors $\{|a\rangle, |b\rangle, |c\rangle\}$ form an orthonormal basis for $\mathcal{H}$ where some operator $\hat{K}$ lives. Suppose we know also that these vectors are *eigenvectors* of $\hat{K}$:

$$\hat{K}|a\rangle = 5|a\rangle \quad \hat{K}|b\rangle = -12|b\rangle \quad \hat{K}|c\rangle = 2\mathbf{i}|c\rangle \tag{1}$$

(a) Write the matrix $\hat{K}$ using Dirac notation. Hint: $\hat{K} = a\ |a\rangle\langle a| + b\ |b\rangle\langle b| + c\ |c\rangle\langle c|$

(b) Consider $|\psi\rangle = \frac{1}{\sqrt{2}}(2|a\rangle + 5\mathbf{i}|c\rangle)$ and compute $\langle\psi|\hat{K}|\psi\rangle$.
Do this using Dirac notation and regular matrix multiplication.

4. **Gone with a Trace**

Recall the trace of an operator $\text{Tr}\,[A] = \sum_m \langle m|A|m\rangle$ for the some basis set $\{|m\rangle\}$

Prove that this definition is independent of basis.

Prove the cycle property: $\text{Tr}\,[ABC] = \text{Tr}\,[BCA] = \text{Tr}\,[CAB]$

Note that this implies if A is diagonalizable with eigenvalues λ_i that $\text{Tr}\,[A] = \sum_i \lambda_i$

5. **A Very Useful Trick**

Consider an operator A. Show the following identity

$$\det e^A = e^{\text{Tr}\,[A]} \tag{2}$$

Hint: Recall that the determinant is the product of eigenvalues

6. **Another Useful Trick**

Recall that for generic operators A and B that $e^A e^B \neq e^{A+B}$ but rather

$$Z(A,B) \equiv \log(e^A e^B) = A + B + \frac{1}{2}[A,B] + \cdots \tag{3}$$

where the $\cdots$ are terms involving commutators of commutators. Ick. This is known as the BCH expansion. Note the possibility to simplify.

One formal expression for (3) is

$$Z(A,B) = A + B + \int_0^1 \mathrm{d}t \sum_{n=1}^{\infty} \frac{(\mathbb{1} - e^{L_A} e^{tL_B})^{n-1}}{n(n+1)} \frac{(e^{L_A} - \mathbb{1})}{L_A}[A,B] \tag{4}$$

where $L_X Y \equiv [X,Y]$

Consider $[A,B] = uA + vB + c\mathbb{1}$.

(a) Show that $L_A[A,B] = v[A,B]$ and $L_B[A,B] = -u[A,B]$

(b) Simplify (4) using the above eigenvalue relations.

(c) Evaluate the integral/sum explicitly. Use $\frac{\log(x)x}{x-1} = 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}$

You should get $Z(A,B) = A + B + f(u,v)[A,B]$ where

$$f(u,v) = \frac{(u-v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)} \tag{5}$$