University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2016
Assignment 1

Due 11am Thursday, September 29, 2016

Please note the correction to the due date, and also the postponement of the third problem to the second homework set. Thanks to those of you who have helped improve my factors of two.

1. **Heisenberg picture fields.**

   Here we will try to understand in what sense the field momentum of a free scalar field is $\pi \sim \dot{\phi}$, and we will explain the factor of $\mathbf{i}\omega$ by which π and ϕ differ.

   I usually think in what is called *Schrödinger picture*, where we evolve the states in time

$$|\psi(t)\rangle = \mathbf{U}(t)^\dagger |\psi(0)\rangle = e^{-\mathbf{i}\mathbf{H}t/\hbar} |\psi(0)\rangle$$

   and leave the operators alone. It is sometimes useful to define time-dependent operators by implementing the change of basis associated with $\mathbf{U}$ on the operators[1]:

$$\mathbf{A}(t) \equiv \mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger = e^{+\mathbf{i}\mathbf{H}t/\hbar}\mathbf{A}e^{-\mathbf{i}\mathbf{H}t/\hbar}.$$

   First consider a simple harmonic oscillator,

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

   with

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a} + \mathbf{a}^\dagger\right) = \sqrt{\frac{\hbar}{2m\omega}}2\mathrm{Re}\left(\mathbf{a}\right).$$

$$\mathbf{p} = -\mathbf{i}\sqrt{\frac{\hbar m\omega}{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right) = \sqrt{\frac{m\hbar\omega}{2}}2\mathrm{Im}\left(\mathbf{a}\right).$$

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[1] Recall that the signs are designed so that matrix elements are the same in either picture:

$$\underbrace{\left(\langle\phi(0)|\,\mathbf{U}(t)\right)}_{=\langle\phi(t)|}\mathbf{A}\underbrace{\left(\mathbf{U}(t)^\dagger|\psi(0)\rangle\right)}_{|\psi(t)\rangle} = \langle\phi(0)|\underbrace{\left(\mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger\right)}_{\mathbf{A}(t)}|\psi(0)\rangle$$

and the two pictures differ just by moving around parentheses. (Thanks to Forrest Sheldon for the reminder.)

(a) Using the algebra satisfied by $\mathbf{H}$ and $\mathbf{a}$, show that

$$\mathbf{q}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{q}e^{-\mathbf{i}\mathbf{H}t} = \sqrt{\frac{\hbar}{2m\omega}} 2\mathrm{Re}\left(e^{-\mathbf{i}\omega t}\mathbf{a}\right).$$

(b) Using the expression above, show that

$$\mathbf{p}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{p}e^{-\mathbf{i}\mathbf{H}t} = m\partial_t \mathbf{q}(t)$$

in agreement with what you would want from the Lagrangian formulation and from classical mechanics.

The above was pretty simple, I hope. Now we consider a scalar quantum field theory, in say $d+1 = 1+1$ dimensions:

$$\mathbf{H} = \int d^d x \left(\frac{\boldsymbol{\pi}(x)^2}{2\mu} + \frac{1}{2}\mu v_s^2 \left(\vec{\nabla}\boldsymbol{\phi} \cdot \vec{\nabla}\boldsymbol{\phi}\right) + \frac{1}{2} m^2 \boldsymbol{\phi}^2 \right) = \sum_k \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right).$$

$$\boldsymbol{\phi}(x) = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger \right),$$

$$\boldsymbol{\pi}(x) = \frac{1}{\mathbf{i}} \sum_k \sqrt{\frac{\hbar\mu\omega_k}{2}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k - e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger \right),$$

(c) Find ω_k. (Note that I've added a mass term $m^2\phi^2$, relative to the model we studied in lecture. This is why I use μ instead of m for the object which looks like the inertial mass.)

(d) Do a Legendre transformation to construct the action, $S[\phi] = \int dt d^d x \mathcal{L}(\phi, \dot{\phi})$.

(e) Show that

$$\boldsymbol{\phi}(t,x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\phi}(x)e^{-\mathbf{i}\mathbf{H}t} = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega_k t}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x} + \mathbf{i}\omega_k t}\mathbf{a}_k^\dagger \right)$$

(f) Using the previous result, show that

$$\boldsymbol{\pi}(t,x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\pi}(x)e^{-\mathbf{i}\mathbf{H}t} = \mu\partial_t\boldsymbol{\phi}(t,x)$$

so that all is right with the world.

2. **Complex scalar field and antiparticles**

[This problem is related to Peskin problem 2.2.] So far we've discussed scalar field theory with one *real* scalar field. The particles created by this field are their own antiparticles.

To understand this statement better, consider a scalar field theory in $d+1$ dimensions with *two* real fields ϕ_1, ϕ_2. Organize them into one complex field $\Phi \equiv \phi_1 + \mathbf{i}\phi_2$, with $\Phi^\star = \phi_1 - \mathbf{i}\phi_2$, and let

$$S[\Phi, \Phi^\star] = \int d^d x dt \left(\frac{1}{2}\mu \partial_t \Phi \partial_t \Phi^\star - \frac{1}{2}\mu v^2 \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star - V(\Phi^\star\Phi) \right) .$$

(a) Show that

$$S[\Phi, \Phi^\star] = \int \left(\sum_{i=1,2} \left(\frac{1}{2}\mu \left(\partial_t \phi_i\right)^2 - \frac{1}{2}\mu v^2 \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_i \right) - V\left(\phi_1^2 + \phi_2^2\right) \right) .$$

That is, if $V(q^2) = \frac{1}{2}m^2 q^2$, the action is just the sum of two copies of the action of the theory we considered previously.

(b) Show by doing the Legendre transformation that the associated hamiltonian is

$$\mathbf{H} = \int d^d x \left(\frac{2}{\mu}\Pi\Pi^\star + \frac{1}{2}\mu v^2 \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^\star + V\left(\Phi\Phi^\star\right) \right)$$

where the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2}\mu \dot{\Phi}^\star, \quad \Pi^\star = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\star} = \frac{1}{2}\mu \dot{\Phi}$$

with the Lagrangian density $\mathcal{L}$ defined by $S = \int dt d^d x \mathcal{L}$.

(c) This theory has a continuous symmetry under which $\Phi \to e^{\mathbf{i}\alpha}\Phi, \Phi^\star \to e^{-\mathbf{i}\alpha}\Phi^\star$ with α a real constant. Show that the action S does not change if I make this replacement. [2]

(d) The existence of a continuous symmetry means a conserved charge – a hermitian operator which commutes with the Hamiltonian, which generates the symmetry (this is the Emmy "Quantum" Nöther theorem). Show that

$$\mathbf{q} \equiv \int d^d x \, \mathbf{i} \left(\Phi^\star \Pi^\star - \Pi \Phi\right)$$

[2] This is called a U(1) symmetry: it is a unitary rotation (hence 'U') on a one-dimensional (hence '(1)') complex vector. Notice that on the real components ϕ_1, ϕ_2 it acts as a two-dimensional rotation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} .$$

The name for this group is SO(2). So U(1) is the same as SO(2).

generates this transformation, in the sense that the change in the field under a transformation with infinitesimal α is

$$\delta\Phi = \mathbf{i}\alpha\Phi = -\mathbf{i}\alpha[\mathbf{q}, \Phi], \quad \text{and}\delta\Phi^\star = -\mathbf{i}\alpha\Phi^\star = -\mathbf{i}\alpha[\mathbf{q}, \Phi^\star].$$

Show that $[\mathbf{q}, \mathbf{H}] = 0$.

(e) For the case where $V(\Phi\Phi^\star) = m^2\Phi\Phi^\star$ the hamiltonian is quadratic. Diagonalize it in terms of *two sets* of creation operators and annihilation operators. You should find something of the form

$$\Phi = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2\mu}}\sum_k \frac{1}{\sqrt{\omega_k}}\left(e^{\mathbf{i}kx}\mathbf{a}_k + e^{-\mathbf{i}kx}\mathbf{b}_k^\dagger\right). \tag{1}$$

(f) Write the canonical commutators

$$[\Phi(x), \Pi(x')] = \mathbf{i}\hbar\delta(x - x'), \qquad [\Phi(x), \Pi^\star(x')] = 0$$

(and the hermitian conjugate expressions) in terms of $\mathbf{a}$ and $\mathbf{b}$.

(g) Rewrite $\mathbf{q}$ in terms of the mode operators.

(h) Evaluate the charge of each type of particle created by $\mathbf{a}_k^\dagger$ and $\mathbf{b}_k^\dagger$ (*i.e.* find $[\mathbf{q}, \mathbf{a}^\dagger]$).

I claim that the particle created by $\mathbf{a}^\dagger$ is the antiparticle of that created by $\mathbf{b}^\dagger$ in the sense that they have opposite quantum numbers. This means that we can add terms to the hamiltonian by which they can annihilate each other, without breaking any symmetries. What might such a term look like?