University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2016
Assignment 4

Due 11am Tuesday, October 25, 2016

1. **Non-Abelian currents.**

   In the previous two homeworks, we studied a complex scalar field. Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with

   $$S[\Phi_\alpha] = \int d^d x dt \left(\frac{1}{2} \partial_\mu \Phi_\alpha^\star \partial^\mu \Phi_\alpha - V\left(\Phi_\alpha^\star \Phi_\alpha\right) \right)$$

   Consider the objects

   $$Q^i \equiv \int d^d x \mathbf{i} \left(\Pi_\alpha^\dagger \sigma^i_{\alpha\beta} \Phi_\beta^\dagger \right) + h.c.$$

   where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

   (a) What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of S.

   (b) If you want to, show that $[Q^i, H] = 0$, where H is the Hamiltonian.

   (c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.

   (d) To complete the circle, find the the Noether currents J^i_μ associated to the symmetry transformations you found in part 1a.

   (e) Generalize to the case of N scalar fields.

2. **Recovering non-relativistic quantum mechanics.**

   Consider a complex scalar field, in the non-relativistic limit,

   $$\Phi = \sqrt{2m} e^{-\mathbf{i}mt} \Psi, \quad |\dot{\Psi}| \ll m\Psi.$$

   Recall that in this limit, the antiparticles disappear and the mode expansion is

   $$\Psi(x) = \int đ^d p \, e^{-\mathbf{i}\vec{p}\cdot\vec{x}} \mathbf{a}_p, \quad \Psi^\dagger(x) = \int đ^d p \, e^{\mathbf{i}\vec{p}\cdot\vec{x}} \mathbf{a}_p^\dagger \, .$$

   (a) Show that

   $$\hat{P}_i \equiv \int đ^d p p_i \mathbf{a}_p^\dagger \mathbf{a}_p$$

   is the generator of translations and commutes with the Hamiltonian.

(b) Let

$$\hat{X}^i \equiv \int d^d x \Psi^\dagger(x) x^i \Psi(x).$$

A state of one particle at location $\vec{x}$ is

$$|x\rangle = \Psi^\dagger(x) |0\rangle .$$

Show that

$$\hat{X}^i |x\rangle = x^i |x\rangle .$$

(c) Consider the general one-particle state

$$|\psi\rangle = \int d^d x \ \psi(x) \Psi^\dagger(x) |0\rangle = \int d^x x \ \psi(x) |x\rangle .$$

Show that

$$\hat{X}^i |\psi\rangle = \int d^d x x^i \psi(x) |x\rangle$$

and (a little more involved)

$$\hat{P}^i |\psi\rangle = \int d^d x \left(-\mathbf{i} \frac{\partial}{\partial x^i} \psi(x) \right) |x\rangle ,$$

which is the usual action of these operators on single-particle wavefunctions $\psi(x)$.