University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215A QFT Fall 2017
# Assignment 2

**Due 11am Thursday, October 14, 2017**

1. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism (with no sources) follows from the action

$$S[A] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)} .$$

Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu ,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2} \left( E^2 + B^2 \right), \quad \vec{S} = \vec{E} \times \vec{B} .$$

2. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in the lecture notes. Consider again the complex scalar field from problem 2 of the first assignment.

   Suppose the potential is

$$V(\Phi^\star \Phi) = g\left(\Phi^\star \Phi - v^2\right)^2$$

   where $g, v$ are constants. The important features of $V$ are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^\star$, so that it preserves the particle-number symmetry generated by $\mathbf{q}$ which was the hero of that problem, and (2) the minimum of $V(x)$ away from $x = 0$.

   Treat the system classically. Write the action $S[\Phi, \Phi^\star]$ in polar coordinates in field space:

$$\Phi(x, t) = \rho e^{\mathbf{i}\theta}$$

   where both $\rho, \theta$ are functions of space and time.

   (a) Consider constant field configurations, and show that minimizing the potential fixes $\rho$ but not the phase $\theta$.

   (b) Compute the mass$^2$ of the $\rho$ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.

   (c) Now ignore the deviations of $\rho$ from its minimum (it's heavy and slow and hard to excite), but continue to treat $\theta$ as a field. Plug the resulting expression

$$\Phi = v e^{\mathbf{i}\theta(x,t)}$$

   into the action. Show that $\theta$ is a massless scalar field.

   (d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on $\theta$?

3. **Gaussian integrals are your friend.**

   (a) Show that

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

   [Hint: square the integral and use polar coordinates.]

   (b) Consider a collection of variables $x_i, i = 1..N$ and a real, symmetric matrix $a_{ij}$. Show that

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

   (Summation convention in effect, as always.)

[Hint: change integration variables to diagonalize $a$. $\det a = \prod a_i$, where $a_i$ are the eigenvalues of $a$.]

(c) I added this problem because it might be helpful for the next problem. In that regard, for any function of the $N$ variables, $f(x)$, let

$$\langle f(x)\rangle \equiv \frac{\int \prod_{i=1}^N dx_i e^{-\frac{1}{2}x_i a_{ij} x_j} f(x)}{Z[J=0]}, \quad Z[J] = \int \prod_{i=1}^N dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i}.$$

Show that

$$\langle x_i x_j\rangle = \partial_{J_i}\partial_{J_j} \log Z[J]|_{J=0} = a_{ij}^{-1}$$

Also, convince yourself that

$$\left\langle e^{J_i x_i}\right\rangle = \frac{Z[J]}{Z[J=0]}.$$

(d) Note that the number $N$ in the previous parts may be infinite. This is really the only path integral we know how to do.

4. **Gaussian identity.**

Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}}\right\rangle = e^{-A(K)\left\langle \mathbf{q}^2\right\rangle}$$

and determine $A(K)$. Here $\langle ...\rangle \equiv \langle 0| ... |0\rangle$. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, better, do it both ways.

This result is useful for the following problem and in many other places.

5. **Zero-phonon process.**

We wish to understand the probability for a photon to hit (our crude model of) a crystalline solid without exciting any vibrational excitations.

Fermi's golden rule says that the probability for a transition from one state of the lattice $|L_i\rangle$ to another $|L_f\rangle$ is proportional to

$$W(L_i \to L_f) = |\langle L_f| \mathbf{H}_\mathrm{L} |L_i\rangle|^2.$$

Here $\mathbf{H}_{\mathrm{L}}$ is the hamiltonian describing the interaction between the photon and an atom in the lattice. For the first parts of the problem, use the following form (to be justified in the last part of the problem):

$$\mathbf{H}_{\mathrm{L}} = Ae^{\mathbf{i}K\mathbf{x}} + h.c. \tag{1}$$

where $\mathbf{x}$ is the (center of mass) position operator of the atom in question; $K$ is a constant (the photon wavenumber), and (for the purposes of the first parts of the problem) $A$ is a constant. $+h.c.$ means 'plus the hermitian conjugate of the preceding stuff'.

(a) Recalling that $\mathbf{x}$ (up to an additive constant) is part of a collection of coupled harmonic oscillators:

$$\mathbf{x} = nx + \mathbf{q}_n$$

evaluate the "vacuum persistence amplitude" $\langle 0|\, \mathbf{H}_{\mathrm{L}} \,|0\rangle$. You will find the result of problem 4 useful.

(b) From the previous calculation, you will find an expression that requires you to sum over wavenumbers. Show that in one spatial dimension, the probability for a zero-phonon transition is of the form

$$P_{\text{Mössbauer}} \propto e^{-\Gamma \ln L}$$

where $L$ is the length of the chain and $\Gamma$ is a function of other variables. Show that this infrared divergence is missing for the analogous model of crystalline solids with more than one spatial dimension. (Cultural remark: these amplitudes are called 'Debye-Waller factors').

(c) Convince yourself that a coupling $\mathbf{H}_{\mathrm{L}}$ of the form (1) arises from the minimal coupling of the electromagnetic field to the constituent charges of the atom, after accounting for the transition made by the radiation field when the photon is absorbed by the atom. 'Minimal coupling' means replacing the momentum operator of the atom $\mathbf{p}$, with the gauge-invariant combination $\mathbf{p} \to \mathbf{p} + \mathbf{A}$. You will also need to recall the form of the quantized electromagnetic field in terms creation and annihilation operators for a photon of definite momentum $K$.