1. **Non-Abelian currents.** In previous homework, we studied a complex scalar field. Now, we make a big leap to two complex scalar fields, $\Phi_{\alpha=1,2}$, with

$$S[\Phi_{\alpha}] = \int d^d x dt \left( \frac{1}{2} \partial_\mu \Phi^*_\alpha \partial^\mu \Phi_\alpha - V (\Phi^*_\alpha \Phi_\alpha) \right)$$

Consider the objects

$$Q^i = \frac{1}{2} \int d^d x \left( \Pi^i_{\alpha} \Phi^*_\alpha \Phi^*_\beta \sigma^{i}_{\alpha\beta} + h.c. \right)$$

where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

(a) What symmetries do these charges generate (i.e. how do the fields transform)? Show that they are symmetries of $S$.

(b) If you want to, show that $[Q^i, H] = 0$, where $H$ is the Hamiltonian.

(c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.

(d) To complete the circle, find the Noether currents $J^i_\mu$ associated to the symmetry transformations you found in part 1a.

(e) Generalize to the case of $N$ scalar fields.

2. **More about 0+0d field theory.** Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

Consider the case where we put a label on the field: $q \rightarrow q_a, a = 1..N$. So we are studying

$$Z = \int \prod_a dq_a e^{-S(q)}$$

Let

$$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

where $T_{abcd}$ is a collection of couplings. Assume $K_{ab}$ is a real symmetric matrix.
(a) Show that the propagator has the form:

\[ a \rightarrow b = (K^{-1})_{ab} = \sum_k \phi_a(k) \ast \frac{1}{k} \phi_b(k) \]

where \( \{k\} \) are the eigenvalues of the matrix \( K \) and \( \phi_a(k) \) are the eigenvectors in the \( a \)-basis.

(b) Show that in a diagram with a loop, we must sum over the eigenvalue label \( k \). (For definiteness, consider the order-\( g \) correction to the propagator.)

(c) Consider the case where \( K_{ab} = t (\delta_{a,b+1} + \delta_{a+1,b}) \), with periodic boundary conditions: \( a + N \equiv a \). Find the eigenvalues. Show that in this case if

\[ T_{abcd}q_a q_b q_c q_d = \sum_a g q_a^4 \]

the \( k \)-label is conserved at vertices, i.e. the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

(d) (Bonus question) What is the more general condition on \( T_{abcd} \) in order that the \( k \)-label is conserved at vertices?

(e) (Bonus question) Study the physics of the model described in 2c.

Back to the case without labels.

(f) By a change of integration variable show that

\[ Z = \int_{-\infty}^{\infty} dq \ e^{-S(q)} \]

with \( S(q) = \frac{1}{2} m^2 q^2 + g q^4 \) is of the form

\[ Z = \frac{1}{\sqrt{m^2}} \mathcal{Z} \left( \frac{g}{m^4} \right) \]

This means you can make your life easier by setting \( m = 1 \), without loss of generality.

(g) Convince yourself (e.g. with Mathematica) that the integral really is expressible as a Bessel function.

(h) It would be nice to find a better understanding for why the partition function of (0+0)-dimensional \( \phi^4 \) theory is a Bessel function. Then find a Schwinger-Dyson equation for this system which has the form of Bessel’s equation for

\[ K(y) \equiv \frac{1}{\sqrt{y}} e^{-a/y} \mathcal{Z} (1/y) \]

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for some constant $a$. (Hint: I found it more convenient to set $g = 1$ for this part and use $\xi \equiv m^2$ as the argument. If you get stuck I can tell you what function to choose for the ‘anything’ in the S-D equation.)

(i) Make a plot of the perturbative approximations to the ‘Green function’ $G \equiv \langle q^2 \rangle$ as a function of $g$, truncated at orders 1 through 6 or so. Plot them against the exact answer.

(j) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large $n$ (by brute force or by cleverness).

3. **Combinatorics from 0-dimensional QFT.** [This is a bonus problem.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is $C_n$ or $C_{n+1}$).

One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields $h$ and $l$.
- There is an $\sqrt{t}h^2l$ vertex in terms of a coupling $t$.
- The bare $l$ propagator is 1.
- The bare $h$ propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.\(^1\)
- There are no loops of $h$.

The last two rules can be realized from a lagrangian by introducing a large $N$ (below).

(a) Show that the full two-point green’s function for $h$ is

$$G(t) = \sum_n t^n C_n$$

\(^1\)An annoying extra rule: All the $l$ propagators must be on one side of the $h$ propagators. You’ll see in part 3f how to justify this.
the generating function of Catalan numbers.

(b) Let $\Sigma(t)$ be the sum of diagrams with two $h$ lines sticking out which may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and $\Sigma$ is called the “1PI self-energy of $h$”. We’ll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1 - \Sigma(t)}$.

(c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)

\[
\Sigma = G(t) \tag{1}
\]

where $\Sigma$ is the 1PI self-energy of $h$.

(d) Solve this equation for the generating function $G(t)$.

(e) If you are feeling ambitious, add another coupling $N^{-1}$ which counts the crossings of the $l$ propagators. The resulting numbers can be called Touchard-Riordan numbers.

(f) How to realize the no-crossings rule? Consider

\[
L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha \beta} h_{\alpha} h_{\beta} + \sum_{\alpha, \beta} l_{\alpha \beta}^2 + \sum_{\alpha} h_{\alpha}^2
\]

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large $N$ are the ones we kept above. Hint: to keep track of the index loops, introduce (‘t Hooft’s) double-line notation: since $l$ is a matrix, it’s propagator looks like: $\beta \cdots \alpha$, while the $h$ propagator is just one index line $\alpha \cdots \alpha$, and the vertex is $\circ$. If you don’t like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha \beta} l_{\alpha \beta} \rangle = \overset{\alpha}{\circ} \cdots \overset{\alpha}{\circ}$, $\langle h_{\alpha} h_{\alpha} \rangle = \overset{\alpha}{\circ} \cdots \overset{\alpha}{\circ}$ and the $hhl$ vertex is: $\overset{\alpha}{\circ} \overset{\alpha}{\circ} \overset{\alpha}{\circ}$.

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.
(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.

4. Brain-warmer: the identity does nothing twice. Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:

\[ \mathbb{1}_1^2 = \mathbb{1}_1 = \int \frac{d^d p}{2\omega_p} |\vec{p}\rangle \langle \vec{p}|. \]