University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2017
Assignment 8

Due 11am Tuesday, November 28, 2017

1. **Brain-warmer: Scalar particle scattering cross-sections.**

   What is the leading-order differential cross-section $\frac{d\sigma}{d\Omega}$ for $2 \to 2$ snucleon-snucleon scattering in $d = 3$ space dimensions in the center-of-mass frame?

   What is the total cross section in the limit that the snucleons are massless?

2. **Lorentz algebra.**

   (a) Check that the $\mathsf{so}(1,3)$ algebra indeed factorizes into $\mathsf{su}(2)_L \times \mathsf{su}(2)_R$ when written in terms of $\vec{J}^{\pm} = \frac{1}{2}\left(\vec{J} \pm \mathbf{i}\vec{K}\right)$.

   (b) Show that given a collection of k matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (the Clifford algebra), we can make a k-dimensional representation of $SO(1,d)$ with generators
   $$J^{\mu\nu} = \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu].$$
   As an intermediate step, it is helpful to show that
   $$[J^{\mu\nu}, \gamma^\rho] = \gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}.$$
   Convince yourself that this last equation says that γ^μ transforms as a four-vector, *i.e.*
   $$[\gamma^\mu, J^{\mu\nu}_{\text{Dirac}}] = (J^{\rho\sigma}_{\text{vector}})^\mu{}_\nu \gamma^\nu.$$

   (c) Given a left-handed Weyl spinor ξ_L and a right-handed Weyl spinor Ψ_R, show that the combination
   $$\xi_L \sigma^\mu \psi_R$$
   transforms as a four-vector. Show that
   $$\psi_R \bar{\sigma}^\mu \xi_L$$
   does too.

3. **Brain warmer: $\mathbb{Z}_2$ symmetry of real scalar field theory.**

   What does the operator
   $$U \equiv e^{\mathbf{i}\pi \sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}$$
   do to the real scalar field
   $$\phi(x) \to \phi'(x) = U\phi(x)U^\dagger$$
   whose ladder operators are $\mathbf{a}, \mathbf{a}^\dagger$?

   For which Lagrangians is this a symmetry?

4. **Charge conjugation in complex scalar field theory.**

   Consider again a free complex Klein-Gordon field Φ. Define a discrete symmetry operation (charge conjugation) C, by
   $$\Phi(x) \mapsto C\Phi(x)C^{-1} = \eta_c \Phi^\dagger(x)$$
   where C is a unitary operator, and η_c is an arbitrary phase factor. Assume that the vacuum is invariant under charge conjugation: $C|0\rangle = |0\rangle$.

   (a) Show that the free lagrangian is invariant under C , but the particle number current j^μ changes sign.

   (b) Show that the annihilation operators satisfy
   $$C\mathbf{a}_k C^{-1} = \eta_c \mathbf{b}_k, \quad C\mathbf{b}_k C^{-1} = \eta_c^\star \mathbf{a}_k$$
   and hence show that C interchanges particle and antiparticle states, up to a phase.

5. **Parity symmetry of real scalar field theory.**

   Under the parity transformation
   $$\vec{x} \mapsto \vec{x}' = -\vec{x}$$
   a real Klein-Gordon transforms as
   $$\phi(t, \vec{x}) \mapsto P\phi(t, x)P^{-1} = \eta_p \phi(t, -\vec{x}) \tag{1}$$
   where P is unitary and $\eta_P = \pm 1$ is the *intrinsic parity* of the field ϕ. Again assume $P|0\rangle = |0\rangle$.

   (a) Show that the parity transformation preserves the free action, for both values of η_P.

(b) Show that an arbitrary n-particle state transforms as

$$P\left|\vec{k}_1, \cdots \vec{k}_n\right\rangle = \eta_P^n \left|-\vec{k}_1, \cdots, -\vec{k}_n\right\rangle.$$

(c) Here we give an explicit realization of the parity operator. Let

$$P_1 \equiv e^{-\mathbf{i}\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}, \quad P_2 \equiv e^{\mathbf{i}\eta_p \frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_{-k}}.$$

Show that

$$P_1 \mathbf{a}_k P_1^{-1} = \mathbf{i}\mathbf{a}_k, \quad P_2 \mathbf{a}_k P_2^{-1} = -\mathbf{i}\eta_p \mathbf{a}_{-k}.$$

Hint: Use the following version of the Campbell-Baker-Hausdorff formula

$$e^{\mathbf{i}\alpha A} B e^{-\mathbf{i}\alpha A} = \sum_{n=0}^{\infty} \frac{(\mathbf{i}\alpha)^n}{n!} B_n$$

where $B_0 \equiv B$ and $B_n = [A, B_{n-1}]$ for $n = 1, 2.....$

Show that $P \equiv P_1 P_2$ is unitary, and satisfies (1).

6. **Action on the current.**

Consider a complex scalar field. Using the results from problems 4 and 5, find the action of parity on the particle current $j^\mu \mapsto P j^\mu P^{-1}$ and $j^\mu \mapsto C j^\mu C^{-1}$. (You'll have to extend the action of P from the case of a real field to the complex case.)

7. **Other bases for gamma matrices.**

Many different bases of gamma matrices are frequently used by humans. You may read on the internet someone telling you that the gamma matrices are

$$\tilde{\gamma}^0 = \begin{pmatrix} 1\!\!1_{2\times 2} & 0 \\ 0 & -1\!\!1_{2\times 2} \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and think that I have lied to you. This basis is useful for studying the non-relativistic limit. The Weyl basis which we introduced in lecture makes manifest the reducibility of the Dirac spinor into L plus R Weyl spinors. Find the unitary matrix U which relates them $\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger$.