University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2017
Assignment 9

Due 11am Tuesday, December 5, 2017

1. **Brain-warmer.** Check that

$$(p \cdot \sigma)\,(p \cdot \bar{\sigma}) = p^2.$$

2. **Symmetries of the Dirac lagrangian.**

   (a) Find the Noether currents j^μ and j_5^μ associated with the transformations $\Psi \to e^{-\mathbf{i}\alpha}\Psi$ and $\Psi \to e^{-\mathbf{i}\alpha\gamma^5}\Psi$ of a free Dirac field. Show by explicit calculation that the former is conserved and the latter is conserved if $m = 0$.

   (b) Find the conserved currents associated with the Lorentz symmetry $\Psi \mapsto \Lambda_{\text{half}}(\theta, \beta)\Psi$ of the Dirac Lagrangian. Show that the conserved charge takes the form mentioned in lecture

$$J^{\mu\nu} = \int_{\text{space}} \left(\mathcal{J}^{\mu\nu}_{\text{orbital}} + \Psi^\dagger J^{\mu\nu}_{\text{Dirac}}\Psi\right)$$

   where $\mathcal{J}^{\mu\nu}_{\text{orbital}}$ has the form it would have for a scalar field, and $J^{\mu\nu}_{\text{Dirac}} \equiv \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]$ are the matrices satisfying the Lorentz algebra. Convince yourself that this matrix specifies how the current acts in the one-particle sector.

3. **Majorana mass.**

   (a) Show that a *majorana mass* term for a Weyl fermion

$$\mathcal{L}_m = m\psi_R^t \mathbf{i}\sigma^2\psi_R + h.c. = m\,(\psi_R)_\alpha\,\epsilon^{\alpha\beta}\,(\psi_R)_\beta + h.c.$$

   is Lorentz invariant, but violates particle number. Figure out what the $+h.c.$ is explicitly.

   (b) Find the equations of motion (don't forget the kinetic term).

   (c) Why isn't $(\psi_R)_\alpha\,\epsilon^{\alpha\beta}\,(\psi_R)_\beta \stackrel{?}{=} 0$, given the antisymmetry under $\alpha \leftrightarrow \beta$?

   (d) Take a 4-component Dirac field $\Psi(x) = \begin{pmatrix}\Psi_L \\ \Psi_R\end{pmatrix}$, and identify $\Psi_L = \chi, \Psi_R = \mathbf{i}\sigma^2\chi^\star$, where χ is a Weyl spinor. Compare with the previous action.

4. **Negative-energy solutions of the Dirac equation.** Check that $\Psi(x) = v(p)e^{+\mathbf{i}p\cdot x}$ with

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \quad s = 1, 2$$

solves the Dirac equation if $p^2 = m^2$ and $p^0 > 0$.

Assuming that η^s comprise an orthonormal basis of 2×2 spinors, check that

$$\sum_{s=1,2} v^s \bar{v}^s = \not{p} - m.$$

Check that $(v^s)^\dagger (p) v^{s'}(p) = 2\omega_p \delta^{ss'}$. (You might want to choose $\vec{p} = \hat{z}p^3$ and a basis of σ^3 eigenstates to do this.)

5. **Supersymmetry.** A continuous symmetry which mixes bosons and fermions is called *supersymmetry*.

   (a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, with Lagrangian is

   $$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi + \chi^\dagger\mathbf{i}\bar{\sigma}^\mu\partial_\mu\chi + F^\star F.$$

   Here F is an auxiliary field whose purpose is to make the supersymmetry transformations look nice. Show that the action is invariant under

   $$\delta\phi = -\mathbf{i}\epsilon^T\sigma^2\chi, \delta\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^\star, \delta F = -\mathbf{i}\epsilon^\dagger\bar{\sigma}\cdot\partial\chi \tag{1}$$

   where the symmetry parameter ϵ is a 2-component spinor of Grassmann numbers.

   (b) Show that the term

   $$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2}\mathbf{i}m\chi^T\sigma^2\chi\right) + h.c.$$

   is also invariant under the transformation (1). Eliminate F from the full Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its equations of motion, and show that the fermion and boson fields are given the same mass.

   (c) We can include supersymmetric interactions as well. Show that the following field theory is supersymmetric:

   $$\mathcal{L} = \partial_\mu\phi_i^\star\partial^\mu\phi^i + \chi_i^\dagger\mathbf{i}\bar{\sigma}\cdot\partial\chi_i + F_i^\star F_i + \left(F_i\partial_{\phi_i}W + \frac{\mathbf{i}}{2}\partial_{\phi_i}\partial_{\phi_j}W\chi_i^T\sigma^2\chi_j + h.c.\right)$$

   where $i = 1..n$ and $W = W(\phi)$ is an arbitrary function of the ϕ_i, called the *superpotential*. For the simple case $n = 1$ and $W = g\phi^3/3$ write out the field equations for ϕ and χ after eliminating F.

6. **The magnetic moment of a Dirac fermion.** [From L. Hall]

In this problem we consider the hamiltonian density

$$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu \Psi A_\mu \ .$$

As we discussed, this describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field Ψ and a vector potential A_μ. In this problem, we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by $\langle 0| \Psi(x) |\vec{p}, s\rangle = e^{-\mathbf{i}px} u^s(p)$. We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term $h_{NR} \ni \mu_B \vec{S} \cdot \vec{B}$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the particle's spin operator.

(a) q is a real number. What is required of A_μ for $H_I = \int d^3x \mathfrak{h}_I$ to be hermitian?

(b) How must A_μ transform under parity P and charge conjugation C in order for H_I to be invariant? How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

(c) Show that in the non-relativistic limit

$$\bar{u}(p')\gamma^{\mu\nu} u'(p) F_{\mu\nu} = a\xi^\dagger \sigma \cdot \vec{B}\xi'$$

for some constant a (find a). Recall that $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Here u, u' are positive-energy solutions of the Dirac equation with mass m and

$$u \overset{NR}{\to} \sqrt{m}(\xi, \xi), u' \overset{NR}{\to} \sqrt{m}(\xi', \xi')$$

in the non-relativistic limit.

(d) Suppose that A_μ describes a magnetic field $\vec{B}$ which is uniform in space and time.

Show that in the non-relativistic limit

$$\langle \vec{p}', s'| H_I |\vec{p}, s\rangle = \not{\delta}^3 (\vec{p} - \vec{p}')\, h(\xi, \xi', \vec{B})$$

and find the function $h(\xi, \xi', \vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.* $\langle \vec{p}|\vec{p}'\rangle_{NR} = \not{\delta}^3(p - p')$). Interpret h as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g\frac{|q|}{2m}$ with $g = 2$.

(e) [optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M}\bar{\Psi} F_{\mu\nu}[\gamma^\mu, \gamma^\nu]\Psi \ \ ?$$