Lecture 23 and CY Geometry Notes

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Reading: GSW ch. 12, 14, 15, 16.

1 Geometry and topology summary

I've stolen this summary from some notes by Shamit Kachru.

1.1 Cohomology and Homology

A p-form is a completely antisymmetric p-index tensor,

$$A \equiv \frac{1}{p!} A_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p}.$$

The *wedge product* of a p-form A and a q-form B is a $p+q$ form with components

$$(A \wedge B)_{m_1 \dots m_{p+q}} = \frac{(p+q)!}{p!q!} A_{[m_1 \dots m_p} B_{m_{p+1} \dots m_{p+q}]}$$

where [..] means sum over permutations with a -1 for odd permutations. The space of p-forms on a manifold X is sometimes denoted $\Omega^p(X)$.

The *exterior derivative* d acts on forms as

$$d : \Omega^p(X) \to \Omega^{p+1}$$

by

$$(dA_p)_{m_1 \dots m_{p+1}} = (p+1) \partial_{[m_1} A_{m_2 \dots m_{p+1}]}.$$

You can check that

$$d^2 = 0.$$

Because of this property $d^2 = 0$, it is possible to define *cohomology* – the image of one $d : \Omega^p \to \Omega^{p+1}$ is in the kernel of the next $d : \Omega^{p+1} \to \Omega^{p+2}$ (*i.e.* the Ω^ps form a chain complex).

A form ω_p is *closed* if it is killed by d: $d\omega_p = 0$.

A form ω_p is *exact* if it is d of something: $\omega_p = d\alpha_{p-1}$.

The p-th de Rham cohomology is defined to be

$$H^p(X) \equiv \frac{\text{closed } p\text{−forms on } X}{\text{exact } p\text{−forms on } X}.$$

$b^p = \dim H^p(X)$ is the pth betti number.

b_p are topological invariants of X, and the euler character χ can be written in terms of them

$$\chi(X) = \sum_{p=0}^{d=\dim X} (-1)^p b^p(X).$$

The Hodge star operator $\star$ maps p-forms to $d-p$ forms:

$$\star : \Omega^p \to \Omega^{d-p}$$

by

$$(\star A_p)_{m_1 \ldots m_{d-p}} = \frac{1}{p!} \epsilon_{m_1 \ldots m_{d-p}}{}^{m_{d-p+1} \ldots m_p} A_{m_{d-p+1} \ldots m_p}.$$

In terms of star and d, the laplacian is

$$\Delta = \star d \star d + d \star d \star = dd^\dagger + dd^\dagger$$

where $d^\dagger$ is the adjoint of d wrt the inner product on p-forms given by

$$\langle A|B\rangle \equiv \int_X A \wedge \star B.$$

You should check that this gives the equation above for Δ, via

$$d^\dagger = \star d \star$$

(up to possible signs and junk).

A p-form ω is called harmonic if $\Delta\omega = 0$.

It's a fact that harmonic p-forms on X are in one-to-one correspondence with elements of $H^p(X)$. This is proved using the *Hodge decomposition theorem* which says that any form ω can be written as

$$\omega_p = dA_{p-1} + d^\dagger B_{p+1} + C_p$$

where C_p is harmonic. See *e.g.* Griffiths and Harris or Wikipedia for more information about this.

Hodge duality maps harmoinic p-forms to harmonic $d-p$ forms, so $b^p = b_{d-p}$.

Next we'll discuss homology. For submanifolds of X we can define a *boundary operator* δ, which takes a submanifold of X into a formal sum of its boundaries, with signs for orientation. A formal sum of submanifolds with boundary is called a *chain*. This operator has $\delta^2 = 0$, too, because the boundary of a boundary is empty. Again we can define a submanifold N to be closed if $\delta N = 0$ (this is the usual notion of a closed submanifold, *i.e.* that it has no boundary); a closed p-dimensional submanifold is called a *p-cycle*. N is exact if $N = \delta M$ for some one-higher-dimensional submanifold $M \subset X$.

The p-th homology group of X is

$$H_p(X) \equiv \frac{\text{closed } p\text{−chains on } X}{\text{exact } p\text{−chains on } X}.$$

An important relation between homology and cohomology arises from integration. There is a one-to-one map

$$H^p(X) \to H_{d-p}(X)$$

defined by

$$A_p \mapsto [A]_{d-p}$$

with the $d-p$-cycle $[A]$ satisfying

$$\int_{[A]} \omega_{d-p} \equiv \int_X A \wedge \omega_{d-p} \qquad \forall \omega_{d-p}.$$

This is the thing that should be called Poincaré duality.

1.2 Complex manifolds

A manifold X_d of even dimension $d = 2n$ is complex if we can coordinatize it by n complex coords z^i with holomorphic transition functions between patches

$$z^i \to \tilde{z}^i(z^j).$$

On such a space, it is natural to define (p,q)-forms

$$\omega_{i_1 \ldots i_p, \bar{j}_1 \ldots \bar{j}_q} \in \Omega^{(p,q)}$$

and write

$$d = \bar{\partial} + \partial$$

with

$$\partial = dz^i \frac{\partial}{\partial z^i}, \quad \bar{\partial} = d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}.$$
$$\partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)}$$

$$\bar{\partial} : \Omega^{(p,q)} \rightarrow \Omega^{(p,q+1)}.$$

It's a fact that

$$\partial^2 = 0 = \bar{\partial}^2$$

and hence we can define *dolbeault cohomology* by

$$H^{p,q}_{\bar{\partial}}(X) \equiv \frac{\bar{\partial}-\text{closed } (p,q)-\text{forms on } X}{\bar{\partial}-\text{exact } (p,q)-\text{forms on } X}.$$

and the hodge numbers of X are

$$h^{p,q}(X) \equiv \dim H^{p,q}_{\bar{\partial}}(X).$$

Note that a given space X many have many inequivalent complex structures. We've already seen this on the worldsheet of the string, in the case of a torus, for example where there is a whole keyhole worth of them.

1.3 kähler manifolds

kähler manifolds are complex manifolds with a hermitean metric $G_{ij} = G_{\bar{i}\bar{j}} = 0$ such that if we define a two-form

$$K \equiv i G_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$$

that two-form is closed, $dK = 0$, and is called the Kähler form.

For Kähler manifolds,

$$H^{p,q}_{\bar{\partial}}(X) = H^{p,q}_{\partial}(X) = H^{p,q}(X)$$

the idea being that in this case the two bits of $d = \bar{\partial} + \partial$ commute with each other and with the laplacian[1] . This implies that $b^r = \sum_{p=0}^{r} h^{p,r-p}$. Also, complex conjugation implies that $h^{p,q} = h^{q,p}$. Hodge star implies $h^{n-p,n-q} = h^{p,q}$.

The fact that K is a closed two form of type $(1,1)$ means that it defines a *kähler class* in $H^{1,1}(X)$. Given a basis $\omega_A, \quad A = 1...h^{1,1}$ for $H^{1,1}$, we can expand K as

$$K = \sum_A t^A \omega_A$$

and the t^A are good coordinates on the K$\acute{}$ahler moduli space of X.

1.4 Calabi-Yau spaces

A CY is a Ricci-flat Kähler manifold. Define the Ricci-form on a Kähler manifold to be

$$\mathcal{R} \equiv R_{i\bar{j}} dz^i d\bar{z}^{\bar{j}};$$

[1] In the context of the susy nonlinear sigma model on such a space this is why there is enhanced supersymmetry when the target is k$\acute{}$ahler.

On a Kähler manifold, this is closed $d\mathcal{R} = 0$. So $\mathcal{R}$ defines an equivalence class in $H^{1,1}$, which is called the first chern class $c_1(TX)$. Clearly this vanishes, $c_1 = 0$ for a CY.

Theorem (Calabi, Yau): For any Kähler manifold X with $c_1(X) = 0$, $\exists$ a unique Ricci-flat metric with any given complex structure and Kähler class.

Theorem: A Kähler manifold X has $c_1(X) = 0 \Leftrightarrow \exists$ a nowhere-vanishing holomorphic (3,0)-form Ω.

On a CY it is a fact that: $h^{p,0} = h^{3-p,0}$ (Serre duality). Further, if the holonomy of X is not a proper subgroup of $SU(3)$, $b_1 = 0 = h^{1,0} = h^{0,1}$ and hence by the previous statement $h^{2,0} = h^{0,2} = 0$.

The *Hodge diamond* of a CY manifold then looks like:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

and the euler character is

$$\chi(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) = h^{0,0} - 2h^{3,0} + h^{3,3} - 2h^{2,1} + 2h^{1,1} = 2\left(h^{1,1} - h^{2,1}\right)$$

2 Introduction

Below is the transcript of the lecture by Esteemed Guest Lecturer, Dr. MC Hawking.

3 Supersymmetric sigma models and the de rham complex

Now, we will have a topological interlude, about supersymmetric sigma models and the de rham complex.

Afterwards we will return to the construction of manifolds that admit calabi-yau metrics. After that we will discuss grand unification from compactification of the heterotic string.

Let's try to use the perspective of the field theory living on a fundamental string to gain some insight into Calabi-Yau compactification.

Consider the quantum field theory on the worldsheet of a string (say a type two string) on minkowski space times a calabi-yau three fold, X. It is a direct product of a free c equals 6 superconformal field theory for minkowski space, and a one comma one supersymmetric nonlinear sigma model on the calabi-yau X.

The latter factor is a theory of maps from the worldsheet to X. We can specify the action governing these maps in local coordinates. If X is Ricci flat, which as we've discussed is a consequence of the Calabi-Yau condition, this 2d quantum field theory is a conformal field theory (up to the same order alpha prime corrections as we saw in the supergravity description).

In the case where X is kahler, the nonlinear sigma model (with the same field content!) has extra supersymmetry, in fact it has two comma two supersymmetry This extra symmetry means extra power.

The action can be written as an integral over all of the four-dimensional superspace of the kahler potential.

$$S_X = \int d^2\sigma \int d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{j}})$$

The coordinates are now chiral superfields of 2d (2,2):

$$\Phi^i = \phi^i(y) + \theta^\alpha \psi^i_\alpha + ...$$

(if necessary, see the susy pset for a reminder). Note that the kahler potential is more or less an unknowable thing.

Why does the (1,1) sigma model on a kahler target have (2,2) supersymmetry? because the action can be written as $\int d^4\theta K$. this thing just *is* the (1,1) sigma model on the kahler manifold, but it manifestly has twice as much supersymmetry. The crucial property of the kahler manifold that we are using is that the kahler metric can be written as the second derivative of a kahler potential.

Let's ask a basic question about these theories. How many supersymmetric ground states does this two d quantum field theory have?

One reason to ask about this is that the susy ground states of a two comma two superconformal field theory turn out to correspond to the possible deformations of the action preserving two comma two superconformal invariance, which correspond to massless scalar moduli fields in spacetime.

for those of you who are late, this is happening because i have lost my voice.

A second important fact is that this data is very robust under variations of the action governing the theory, for example under changes in the metric we put on the space.

To see why this is, recall our discussion of the properties of BPS states. Supersymmetric ground states are a special, simple case of BPS states. The basic point is that the supersymmetry charge annihilates an E equals zero state. But any nonzero energy state is mapped by the supercharge into a state of the same energy and opposite fermion number. This is because the supercharge carries fermion number one, and commutes with the hamiltonian. But this means that in order for the number of supersymmetric ground states to change, they have to pair up, so that one is the image of another under the action of Q .

So the sum over ground states of minus one to the fermion number cannot change. But since nonzero-energy states come in pairs of opposite fermion number, the sum over ALL states of minus one to the fermion number only gets contributions from ground states. If it makes you feel better, you can stick a boltzmann factor in the trace, and it doesn't change – this quantity, called the witten index, is 'temperature independent'. And it's invariant under changes in the metric of a nonlinear sigma model (within reason!), and this means that we can compute it. To summarize,

$$\mathrm{tr}\ _{\text{ground states}}(-1)^F = \mathrm{tr}\ _{\mathcal{H}}(-1)^F = \mathrm{tr}\ _{\mathcal{H}}(-1)^F e^{-\beta H}.$$

Here $\mathcal{H}$ is the whole hilbert space of the susy theory. [2]

I'm talking about this because it will give me an excuse to remind you about de rham cohomology. What I want to say is more general than just for kahler manifolds, so let's consider a general one comma one sigma model in, say, two dimensions (the number of dimensions is also not crucial).

The action can be written as an integral over the now-two-dimensional superspace as follows:

$$S = \int d^2\sigma \int d^2\theta g_{ij}(\phi) D_\alpha \phi^i D_\beta \phi^j \epsilon^{\alpha\beta}$$

$$\phi^i(\sigma,\theta) = \phi^i(\sigma) + \theta^\alpha \psi^i_\alpha ...$$

(Note that i, j are no longer holomorphic indices.) Here the superspace coordinates and the fermion partners of the coordinates on X are two d majorana fermions.

In components the action looks like this:

$$S = \frac{1}{2}\int d^2\sigma \left(g_{ij}(\phi)\left(\partial_a\phi^i\partial^a\phi_j + \bar{\psi}^i i\gamma^a\partial_a\psi^j\right) + \frac{1}{4}R_{ijkl}\bar{\psi}^i\psi^j\bar{\psi}^k\psi^l\right).$$

The third important fact about these ground states is that their wavefunctionals are localized on position-independent configurations. This is just the statement that a configuration can reduce its energy by not varying over space.

This means that we can answer our question by reducing to supersymmetric QUANTUM MECHANICS, i e one-dimensional field theory.

The action is now the following

$$S = \frac{1}{2}L\int dt \left(g_{ij}(\phi)\left(\dot{\phi}^i\dot{\phi}_j + \bar{\psi}^i i\gamma^0\partial_t\psi^j\right) + \frac{1}{4}R_{ijkl}\bar{\psi}^i\psi^j\bar{\psi}^k\psi^l\right).$$

Here, L is the size of the spatial direction which we can take to be a circle with periodic boundary conditions if we like.

[2] This discussion comes from Witten, NP**B202**(1982) 253, section 10.

Pick a basis of gamma matrices where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the majorana condition says that one component of a fermion is the complex conjugate of the other:

$$\psi^i = \begin{pmatrix} \psi^i \\ (\psi^i)^\star \end{pmatrix}.$$

The supersymmetry algebra is

$$Q^2 = 0 = (Q^\star)^2$$

and

$$\{Q, Q^\star\} = H.$$

Quantizing this theory, the canonical momentum of the boson coordinate acts as a derivative on wavefunctions.

$$p_i = g_{ij}\partial_t\phi^j = -i\frac{D}{D\phi^i}.$$

The fermions are as usual gamma matrices, but now curved space gammas, since they anticommute to the curved metric:

$$\{\psi_i, \psi_j\} = 0 = \{\psi_i^\star, \psi_j^\star\}$$
$$\{\psi_i, \psi_j^\star\} = g_{ij}(\phi).$$

In this basis, we can take the psis to be annihilation operators and the psi stars to be creation operators.

Build the fermion hilbert space on an empty state killed by all the psis:

$$|0\rangle : \psi^i|0\rangle \equiv 0,\ \forall i = 1...d \equiv \mathrm{dim}X. \quad |0\rangle = |\underbrace{\downarrow \,....\, \downarrow}\rangle$$

A general state in the hilbert space is then a sum over states of all fermion numbers. This sum goes from zero up to d, the dimension of X.

$$|f\rangle = f(\phi)|0\rangle + f_i(\phi)\psi_i^\star|0\rangle + f_{ij}(\phi)\psi_i^\star\psi_j^\star|0\rangle + ... + f_{i_1...i_d}(\phi)\psi_{i_1}^\star...\psi_{i_d}^\star|0\rangle.$$

the coefficient of the p-fermion state is a completely antisymmetric function of the coordinates, a p-form.

$$f_{ij} = -f_{ji}.$$

the hilbert space of the susy quantum mechanics is then the de rham complex on X, the space of all p-forms, p equals zero up to d.

$$\mathcal{H} = \oplus_{p=0}^d \Omega^p(X).$$

The space of p-forms on X is sometimes called $\Omega^p(X)$. The degree of a form is the number of fermions excited in the state.

The supercharges take the following form:

$$Q = i \sum_i \psi_i^\star p_i, \quad Q^\star = -i \sum_i \psi_i p_i.$$

How does Q act on this hilbert space?

it takes a p-fermion state and adds a fermion.

at the same time it differentiates the coefficient function.

this is exactly the action of the exterior derivative on forms, it is the curl operator.

$$Q|f\rangle = |df\rangle$$

how does Q star act ? it removes a fermion psi i (if it is present) and at the same time differentiates in the i direction.

this is the action of d dagger, the divergence operator.

$$Q^\star|f\rangle = |d^\dagger f\rangle$$

(the dagger means adjoint with respect to the norm on p-forms given by wedging A with star B and integrating. d dagger is therefore star d star.)

the hamiltonian then takes the form of the laplacian on forms.

$$H = QQ^\star + Q^\star Q = dd^\dagger + d^\dagger d = \Delta$$

the zero energy states the correspond to harmonic forms, those annihilated by the laplacian.

the number of harmonic p-forms on X is called the p-th betti number of X, $b^p(X)$.

the same number can be computed as the cohomology of the exterior derivative acting on forms: the dimension of the space of closed forms, modulo exact forms. this equivalence is called hodge's theorem.

so the number of ground states of our system with fermion number p is the p-th betti number.

the witten index is therefore trace minus one to the f equals sum over degrees of minus one to the degree of the form times the betti number. this is the definition of the euler character of X.

$$\text{tr } (-1)^F = \sum_{p=0}^{d} (-1)^p b^p(X) = \chi(X).$$

note that there is a charge conjugation symmetry of this system, which interchanges psis and psi stars. it takes the empty state (the zero form) to the filled state (the d-form or volume). it maps a general p-form to its hodge dual, obtained by contracting with the epsilon tensor.

this implies that the p eth betti number is the same as the d minus p eth betti number. this is called hodge duality or sometimes poincare duality.

In the special case where X is a kahler manifold, there is enhanced supersymmetry. this means that the supercharge can be written as a sum of two supercharges which are each separately conserved.

this implies that the exterior derivative on a kahler manifold can be decomposed in that way, and in fact it can. the two terms are the dolbeault operator and its conjugate.

on kahler manifold, a general form has p holomorphic indices and q antiholomorphic indices. The dolbeault operator adds a holomorphic index.

this leads to a refinement of the de rham cohomology whose dimensions are called hodge numbers.

the betti numbers are sums of hodge numbers.

are there any questions?

should we change the voice?

Fitter.

Happier.

More productive.

4 the quintic is a Calabi-Yau

Now let's return to our discussion of examples of calabi yau manifolds.

4.1 $c_1(X) = 0$ iff hol'c threeform nowhere vanishes.

A lemma I mentioned last time caused some confusion.

I was trying to convince you that the existence of an everywhere nonvanishing holomorphic three form on a complex manifold implies that its first chern class vanishes.

In general, if ϕ is some field on a complex manifold, and ϕ is charged under a $U(1)$ gauge group whose connection has some field strength F_0, we can say that 'ϕ is a section of a line bundle $\mathcal{L}$'

(this is written $\phi \in \Gamma(\mathcal{L})$) whose first chern class is determined by the cohomology class of F_0:

$$c_1(\mathcal{L}) = [F_0] \in H^2(X).$$

A connection on $\mathcal{L}$ (not necessarily the same one as the one that gives F_0) is given in terms of the connection like $\mathcal{A}^\phi \equiv \partial \ln \phi$ [3]. This is true since it transforms the right way under gauge transformations:

$$\phi \to e^{i\lambda}\phi \Longrightarrow \mathcal{A}^\phi \to \mathcal{A} + \partial\lambda.$$

The field strength of this connection is

$$\mathcal{F}^\phi = \bar{\partial}\partial \ln \phi$$

which is not necessarily the same as F_0. By familiar facts about derivatives on logs, this is a delta function at the zero locus of phi (locally it looks like del bar del log z in two dimensions):

$$\mathcal{F}^\phi = \bar{\partial}\partial \ln \phi \sim \delta^2(\phi)$$

But the first chern class doesn't care which connection we use:

$$c_1(\mathcal{L}) = [F_0] = [\mathcal{F}^\phi].$$

So if ϕ has no zeros, then $c_1(\mathcal{L})$ must vanish.

Now let's return to Ω.

the fact that it's completely antisymmetric with three indices that go from one to three means it is proportional to the epsilon tensor:

$$\Omega_{ijk}(z) = \phi(z)\epsilon_{ijk}.$$

The coefficient is a section of the determinant bundle of the (holomorphic) tangent bundle:

$$\phi \in \Gamma(\det TX)$$

(this is the bundle whose transition functions are the determinants of the transition functions of the tangent bundle.)

Our earlier general discussion then means that c_1 of the tangent bundle is zero if omega has no zeros:

$$c_1(TX) \propto c_1(\det TX) = [\delta^2(\phi)].$$

[3] Note that I am using 'holomorphic gauge', where we set $\mathcal{A}_{\bar{z}} = 0$.

4.2 the degree k hypersurface in CP^N.

now let's return to showing that the candidate form on the quintic in p four doesn't vanish or blow up. actually we want to show this for the degree N+1 hypersurface in $\mathcal{P}^N$.

in the coordinate patch where z^{N+1} doesn't vanish, with affine coords

$$x^a = z^a/z^{N+1}, a = 1...N$$

omega can be written as

$$\Omega = \frac{dx^1 \wedge ..dx^{N-1}}{\frac{\partial G}{\partial x^N}(x^1..x^N, 1)}|_{G=0}.$$

possible concerns we might have about this are:

1. does it have a pole where $\partial G/\partial x^N$ vanishes?

2. what's special about x^N?

to resolve these concerns, we use a trick.

the fact that G is constant on X (in particular zero) means that dG annihilates tangent vectors to X.

so restricted to X , we get a relation

$$\frac{dx^m}{\partial G/\partial x^m} = -\frac{dx^N}{\partial G/\partial x^N} + \sum_{i \neq m,N} f_i dx^i.$$

rearranging this relation we can rewrite omega (ignoring terms that vanish because $dx^i \wedge dx^i$ vanishes).

$$\Omega = (-1)^{N-m} \frac{dx^1 \wedge ... \wedge d\hat{x^m} \wedge ...dx^N}{\partial G/\partial x^m}.$$

the hat means that that factor is absent

this means that omega is well-defined. omega is nonsingular as long as partial G partial x are not all zero at any point on X. such a G is said to be transverse.

the remaining concern is that omega might have a pole or zero at z n plus one equals zero.

introduce new affine coordinates, y, in this patch, obtained by dividing by z one.

$$y^b \equiv z^b/z^1 = x^b \left(\frac{z^{N+1}}{z^1}\right), \quad b = 2...N+1.$$

They are related to the old ones by multiplication by a nonzero complex number. So for G of degree k, the homogeneity property implies this equation:

$$G_k(1, y^2 ... y^{N+1}) = G_k(x^1, ... x^N, 1) \left(\frac{z^{N+1}}{z^1} \right)^k .$$

omega in the new patch is

$$\Omega = (-1)^{1-m} \left(\frac{z^{N+1}}{z^1} \right)^{N+1-k} \frac{dy^2 \wedge ... \wedge \hat{dy^m} \wedge ... dy^{N+1}}{\partial G / \partial y^m}.$$

this is related to the form we would have written directly in the other patch by a factor with zeros, raised to a power n plus 1 minus k.

this means that if and only if

$$0 = N + 1 - k$$

Omega has no pole or zero on X.

this is what we wanted to show.

so in particular to get a three dimensional thing we should take the degree five hypersurface in CP^4.

A better way to understand this numerology is via the gauged linear sigma model, which also gives access to the mysterious region away from the large volume limit.

i was advertising the existence of a technique that allows one to study what happens away from large volume

it is called the gauged linear sigma model. i don't know if we'll have time to talk about it.

4.3 generalizations of the quintic

For about a week in 1985, people thought that there was only one calabi yau manifold, and sat down to compute the mass of the electron. To save you from any disappointment, I'll mention some generalizations of the quintic hypersurface.

1. weighted projective space is a generalization of C P N where we give the different coordinates different charges under the c star action.

2. we can also consider intersections of hypersurfaces.

with both of these generalizations, the first chern class is the sum of the weights of the coordinates minus the sum of the degrees of the defining equations.

3. we can consider replacing the c star action on the projective space by a non-abelian symmetry. this gives grassmannians and flag manifolds. the study of calabi-yau hypersurfaces in these is in its infancy.

it is not even known if the number of topologically different CY manifolds is finite.

4.4 moduli

returning to the quintic, i have a question for you. which degree five polynomial should we use?

in general, such a thing can be written as a sum of monomials, with complex coefficients.

there are 126 such monomomials.

we can do coordinate redefinitions to get rid of some of them. in particular, rotating the z's by a 5 by 5 matrix of complex numbers doesn't do anything.

so there is a 126 - 25 = 101 complex dimensional moduli space of complex structures on the quintic.

in general the dimension of the complex structure moduli space is h two one of X.

$$\dim \mathcal{M}_{CS}(X) = h^{2,1}(X).$$

they can be thought of as ways to infinitesimally change the holomorphic three form.

there is a *torelli theorem* that I am secretly using here, which says that on a CY-manifold, the complex structure data is completely determined by Ω, and hence, given our formula for Ω in terms of G, by G; in particular, the data in Ω give good coordinates on the CS moduli space.

also, the calabi-yau has a size, determined by the size of the projective space into which we stuck it. this size is encoded in the kahler form, which can be varied without changing the fact that the space is calabi-yau.

a general calabi-yau manifold has a kahler moduli space of dimension h one one

$$\dim \mathcal{M}_{kahler}(X) = h^{1,1}(X).$$

5 unification from the heterotic string

Now let's return to the heterotic string on a Calabi-Yau manifold.

We've tried to identify four-dimensional N equals one vacua by demanding that the supersymmetry variations of the fermion fields vanish. We solved the dilatino variation by demanding that the dilaton be constant and the neveu schwarz three form vanish, which is not the most general solution.

In this case, the gravitino variation told us that X should have $SU(3)$ holonomy.

For the heterotic string, we still have the gaugino variation to deal with.

It looks like this,

$$\delta\lambda = F_{mn}\Gamma^{mn}\epsilon$$

We can set this to zero if this object $F\Gamma$ is an infinitesimal $SU(3)$ rotation; this will preserve the same epsilon as the holonomy group.

To see what this means, consider the form of the gamma matrices in complex coordinates.

As usual, the holomorphic gammas and antiholomorphic gammas are annihilation operators and creation operators.

Define the "all-down" spinor by demanding that it is killed by all the holomorphic gammas.

In this basis, the gaugino variation takes the form.

$$\delta\lambda = F_{mn}\Gamma^{mn}\epsilon = \left(F_{ij}\Gamma^{ij} + F_{\bar{i}\bar{j}}\Gamma^{\bar{i}\bar{j}} + F_{i\bar{j}}\Gamma^{i\bar{j}}\right)\epsilon$$

Since they map a given epsilon to different fermion-number sectors (they change the fermion number by $\Delta F = -2, 2, 0$ respectively), these three terms must separately vanish.

We might be tempted to solve this by just setting $F = 0$. But we can't do this on a general CY because of the bianchi identity for B:

$$0 = dB \Rightarrow \text{tr } F \wedge F = \text{tr } R \wedge R.$$

So if this curvature form doesn't vanish, we have to have nonzero F.

This is where the notes end, for now.