MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Problem Set 1
Classical Worldsheet Dynamics

Reading: GSW §2.1, Polchinski §1.2-1.4. Try §3.2-3.3.

Due: Friday, September 21, 2007 at 3 PM, in the 8.821 lockbox.

1. **Weyl symmetry.**[1] are A (global) Weyl transformation acts on a field theory coupled to gravity by
$$F(x) \mapsto \lambda^{-W(F)} F(x)$$
where F is a field, and $W(F)$ is a number called the *Weyl weight* of F. W is normalized so that $W(g_{\mu\nu}) = 2$.

   (a) Find $W(\varphi)$ so that
$$S = \int d^D x \sqrt{g} \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$
is Weyl invariant.

   (b) Consider coordinate transformations of the form $x \mapsto x' = \lambda x$, which can be called Einstein-scale transformations. How do φ and $g_{\mu\nu}$ transform?

   (c) Write down the transformation of φ and $g_{\mu\nu}$ under simultaneous Weyl and Einstein-scale transformations. How does the result for φ relate to the engineering dimension you would assign φ in order to make S dimensionless?

2. **Local Weyl Invariance.** Consider a gravity plus matter system in D dimensions
$$S = S_{grav}[g_{\mu\nu}] + S_{matter}[F, g_{\mu\nu}]$$
where S_{matter} is invariant under the *local* Weyl transformation
$$F(x) \mapsto \lambda(x)^{-W[F]} F(x), \quad g_{\mu\nu}(x) \mapsto \lambda^{-2}(x) g_{\mu\nu}(x).$$

[1] Problems 1-4 are due to Marty Halpern.

Using the equations of motion, show that the matter stress tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S_{matter}}{\delta g_{\mu\nu}}$$

is traceless:

$$T^{\mu\nu} g_{\mu\nu} = 0.$$

3. **Symmetries of the Polyakov action.** Recall the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu.$$

Think of this as the action for a system of matter (X) coupled to 2d gravity (γ_{ab}).

(a) Show that the Polyakov action is locally Weyl-invariant.

(b) Obtain an expression for the stress tensor

$$T_{ab} \equiv -\frac{2}{\sqrt{\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}}$$

and check that it is traceless. [You may want to use the relation

$$\delta(\det A) = \det A \ \mathrm{tr}\ (A^{-1}\delta A),$$

where A is any nonsingular matrix, and δA is any small deformation.]

(c) Use the equation of motion for γ_{ab} to show that when evaluated on a solution

$$S_P[\gamma|_{EOM}, X] = S_{Nambu-Goto}[X].$$

4. **Virasoro algebra (no central extension yet).** Consider the classical mechanics of D two-dimensional free bosons X^μ, which enjoy the canonical Poisson bracket

$$\{X^\mu(\sigma), P^\mu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma')$$

with their canonical momenta P^μ. Show that the objects

$$T_{\pm\pm} \equiv \pm\frac{\pi\alpha'}{2} W_\pm^2, \quad W_\pm^\mu \equiv P^\mu \pm \frac{\partial_\sigma X^\mu}{2\pi\alpha'}$$

satisfy two commuting copies of the Virasoro algebra

$$\{T_{\pm\pm}(\sigma), T_{\pm\pm}(\sigma'\} = (T_{\pm\pm}(\sigma) + T_{\pm\pm}(\sigma'))\partial_\sigma\delta(\sigma - \sigma')$$

$$\{T_{\pm\pm}(\sigma), T_{\mp\mp}(\sigma')\} = 0.$$

You may use the identity

$$f(\sigma)f(\sigma')\partial_\sigma \delta(\sigma - \sigma') = \frac{1}{2}(f^2(\sigma) + f^2(\sigma'))\partial_\sigma \delta(\sigma - \sigma'), \forall f.$$

[This problem may be an opportunity to apply the Tedious Problem Rule discussed in class.]

5. **Spinning strings in AdS.** [This problem might be hard.[2]] In this problem we are going to think about the behavior of a string propagating in 5d anti de Sitter space (AdS). Specifically, we are going to study and use some of its conserved charges. When these conserved quantities are large, they can be compared to their counterparts in a dual 4d gauge theory.

   (a) Write down the Nambu-Goto action for a bosonic string propagating in AdS_5, whose metric (in so-called global coordinates) is

$$ds^2 = R^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2) \equiv G_{ij} dX^i dX^j \qquad (1)$$

   where $d\Omega_3^2$ denotes the line element on the unit three-sphere,

$$d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 d\phi^2).$$

   (b) The AdS background has many isometries. We will focus on two: shifts of t (the energy) and shifts of ϕ (the spin). These isometries of the target space are symmetries of the NLSM, and therefore lead to conserved charges. Using the Noether method, write an expression for the conserved charge S which follows from the symmetry $\phi \to \phi + \epsilon$, with ϵ a constant. Write an expression for the conserved charge E which follows from the symmetry $t \to t + \delta$, with δ a constant.

   We're going to consider a spinning folded string, which spins as a rigid rod around its center, and lies on an equator of the S^3 in (1), $\theta_1 = \theta_2 = \pi/2$. The center of the string is at the 'center' of AdS, $\rho = 0$. Go to a static gauge $t = \tau$, with ρ some function of σ. Consider an ansatz for the azimuthal coordinate

$$\phi = \omega t;$$

[2] It is taken from a recent paper. For privacy's sake, let's call the authors Steve G., Igor K, and Alexandre P. No, that's too obvious. Uhh... let's say S. Gubser, I. Klebanov and A. Polyakov.

this describes a string spinning around the spatial sphere.

(c) Show that with these specifications the Nambu-Goto Lagrangian becomes

$$L = -4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\sqrt{\cosh^2\rho - (\dot{\phi})^2\sinh^2\rho},$$

where $\coth^2\rho_0 = \omega^2$, and the factor of 4 is because there are four segments of the string stretching from 0 to ρ_0.

(d) Show that the energy and spin of this configuration are

$$E = 4\frac{\sqrt{\lambda}}{2\pi}\int_0^{\rho_0} d\rho\frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

$$S = 4\frac{\sqrt{\lambda}}{2\pi}\int_0^{\rho_0} d\rho\frac{\omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

where

$$\lambda = \frac{R^4}{\alpha'^2}.$$

(e) Next we're going to reproduce this solution from the Polyakov action, in conformal gauge:

$$S_P = \frac{1}{4\pi\alpha'}\int d\tau d\sigma G_{ij}\partial_a X^i\partial^a X^j.$$

In this description, a solution must also satisfy the equations of motion following from varying the worldsheet metric, namely the Virasoro constraints:

$$0 = T_{++} = \partial_+ X^i\partial_+ X^j G_{ij}$$

$$0 = T_{--} = \partial_- X^i\partial_- X^j G_{ij}$$

Show that inserting

$$t = e\tau, \phi = e\omega\tau, \rho = \rho(\sigma)$$

into the Virasoro constraints gives

$$(\rho')^2 = e^2(\cosh^2\rho - \omega^2\sinh^2\rho).$$

(Here e is a bit of slop which can be adjusted to make sure period of σ is 2π.) This leads to an expression for $\frac{d\sigma}{d\rho}$.

Show that the (target-space) energy and spin are

$$E = \frac{R^2}{2\pi\alpha'} e \int_0^{2\pi} d\sigma \cosh^2 \rho$$

$$S = \frac{R^2}{2\pi\alpha'} e\omega \int_0^{2\pi} d\sigma \sinh^2 \rho.$$

Use your expression for $\frac{d\sigma}{d\rho}$ to show that this reproduces the answers found using the Nambu-Goto action.

(f) [**super-bonus challenge**] Expand the integrals for E and S at large spin ($\omega = 1 + 2\eta, \eta \ll 1$), and show that in that regime

$$E - S \sim \sqrt{\lambda} \ln S.$$

The fact that this expression is reminiscent of logarithmic scaling violations in field theory is not a coincidence; the coefficient in front of the log is called the 'cusp anomalous dimension.'

Some other problems you might consider doing are

1. **Polchinski Problem 1.2.** Use the Virasoro constraints to show that the endpoints of an open string (with Neumann boundary conditions) move at the speed of light.

2. **Polchinski Problem 1.1.**