MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Problem Set 4

String scattering, open strings, supersymmetry warmup.

Reading: Polchinski, Chapter 6. Please ask about supersymmetry refs.

Due: Thursday, October 25, 2007 at 11:00 AM, in lecture or in the box.

1. **Another tree amplitude (Polchinski 6.11).**

   (a) For what values of $\zeta_{\mu,\nu}$ and k^μ does the vertex operator

   $$\mathcal{O}_\zeta(k) = c\tilde{c}g_c'\zeta_{\mu\nu}(k) : \partial X^\mu \bar{\partial} X^\nu \; e^{ik\cdot X} :$$

   create a physical state of the bosonic string ?

   (b) Compute the three-point scattering amplitude for a massless closed string and two closed-string tachyons at tree level.

   (c) Factorize the tachyon four-point amplitude on the massless pole (in say the s-channel), and use unitarity to relate the massless coupling g_c' to g_c, the coupling for the tachyon.

2. **High-energy scattering in string theory.**[1]

   Consider the tree-level scattering amplitude of N bosonic string states, with momenta k_i^μ:

   $$\mathcal{A}^{(n)}(\{k_1, ..., k_n\}) = \int \prod_{i=4}^{n} d^2 z_i \left\langle \prod_{i=1}^{n} \mathcal{V}_{k_i}(z_i, \bar{z}_i) \right\rangle_{S^2} .$$

   In this problem we will study the limit of hard scattering (also called fixed-angle scattering), where we scale up all the momenta uniformly,

   $$k_i^\mu \mapsto \alpha k_i^\mu, \quad \alpha \to \infty;$$

   in terms of the Lorentz-invariant Mandelstam variables, we are taking the limit $s_{ij} \to \infty$, s_{ij}/s_{kl} fixed. It was claimed in class that the 4-point function behaves

[1] This discussion follows the papers of Gross and Mende.

in this limit like $e^{-s\alpha' f(\theta)}$ where θ is the scattering angle. In this limit, $k^2 \gg m^2$ (if both aren't zero) and we can ignore the parts of the vertex operators other than $e^{ik\cdot X}$.

(a) Notice that the integral over X is always gaussian, hence the action of the saddle point solution gives the exact answer. Find the saddle-point configuration of $X(z)$ and evaluate the on-shell action.

(b) In the hard-scattering limit, the integral over the positions of the $n-3$ unfixed vertex operators is also well-described by a saddle-point approximation. Convince yourself that this is true; *i.e.* that the saddle point is well-peaked.

(c) For the four-point function, find the saddle-point for the z_4 integral and evaluate the action on the saddle. Compare to the claimed behavior of the exact tree-level answer.

3. **Open string boundary conditions.** Polchinski Problems 1.6 and 1.7.

4. **T-duality: not just for the free theory.** Polchinski Problem 8.3.

The following problems are intended to get everyone up to speed with supersymmetry and its consequences. They are more optional than the other ones.

1. **Supersymmetric point particle.**

   Consider the action for a spinning particle

$$S = \int d\tau \left(-\frac{\dot{x}^2}{2e} + e^{-1} i\dot{x}^\mu \psi_\mu \chi - i\psi^\mu \dot{\psi}_\mu \right).$$

   The ψ^μs are real grassmann variables, fermionic analogs of x^μ, which satisfy

$$\psi^\mu \psi^\nu = -\psi^\nu \psi^\mu.$$

   (a) Show that this action is reparametrization invariant, *i.e.*

$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta \psi^\mu = \xi \dot{\psi}^\mu$$

$$\delta e = \partial_\tau(\xi e), \quad \delta\chi = \partial_\tau(\xi\chi)$$

   is a symmetry of S.

   (b) Show that the (local) *supersymmetry* transformation

$$\delta x^\mu = i\psi^\mu \epsilon, \quad \delta e = i\chi\epsilon$$

$$\delta\chi = \dot{\epsilon}, \quad \delta\psi^\mu = \frac{1}{2e}(\dot{x}^\mu + i\chi\psi^\mu)\epsilon.$$

with ϵ a grassmann variable, is a symmetry of S.

(c) Consider the gauge $e = 1, \chi = 0$. What is the equation of motion for χ? What familiar equation for ψ do we get?

2. **Strings in flat space with worldsheet supersymmetry.**

Consider the action for D free bosons and D free fermions in two dimensions:

$$S = -\frac{1}{2\pi}\int d^2\sigma \, \left(\partial_a X^\mu \partial_b X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right).$$

The spacetime $\mu = 0..D-1$ indices are contracted with $\eta_{\mu\nu}$. Here ψ^μ are 2d two-component majorana spinors, and ρ^a are 2d 'gamma' matrices, *i.e.*, they participate in a 2d Clifford algebra $\{\rho^a, \rho^b\} = -2\eta^{ab}$ (we'll work on a Lorentzian worldsheet for this problem), and $\bar{\psi} \equiv \psi^\dagger\rho^0$. Pick a basis for the 2d 'gamma' matrices of the form

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (1)$$

(a) Show that the fermion part of the action above leads to the massless Dirac equation for the worldsheet fermions ψ. Show that with the chosen basis of gamma matrices ρ^α, the Dirac equation implies that $\psi_\pm$ is a function of $\sigma^\pm$ only (where I'm letting the indices on $\rho^a_{\alpha\beta}$ run over $\alpha, \beta = \pm$).

Next we want to show that the (global) *supersymmetry transformation*

$$\delta X^\mu = \bar{\epsilon}\psi^\mu$$

$$\delta\psi^\mu = -i\rho^a\partial_a X^\mu \epsilon$$

is a symmetry of the action S.

(b) A useful first step is to show that for Majorana spinors

$$\bar{\chi}\psi = \bar{\psi}\chi.$$

(c) Show that the action S is supersymmetric.

(d) What is the conserved Noether current $G_{a\alpha}$ associated to the supersymmetry?

(e) [Optional] Show that the algebra enjoyed by the Noether supercharges $Q_\alpha \equiv \int d\sigma G_{0\alpha}$ under Poisson brackets (or canonical (anti)-commutators) is of the form

$$\{Q_\alpha, \overline{Q}_\beta\} = -2i\rho^a_{\alpha\beta}P_a \quad (2),$$

where P_a is the momentum.

Or equivalently, show that the commutator of two supersymmetry transformations (acting on any field) acts as a spacetime translation:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\mathcal{O} = A^a\partial_a\mathcal{O}$$

where A is a constant writeable in terms of $\epsilon_{1,2}$.

(f) Show that the algebra (2) above implies that a state is a supersymmetry singlet ($Q|\psi\rangle = 0$) if and only if it is a ground state of H.

(g) **(1,1) superspace.** Show that the action above can be rewritten as

$$\int d^2z \int d\theta^+ d\theta^- \; D_+\mathbf{X} \cdot D_-\mathbf{X}$$

where $\theta^\pm$ are real grassmann coordinates on $2d, \mathcal{N} = (1,1)$ superspace (*i.e.* there is one real right-moving supercharge and one real left-moving supercharge), and the (1,1) superfield is

$$\mathbf{X}(z, \bar{z}, \theta^+, \theta^-) \equiv X + \theta^-\psi_- + \theta^+\psi_+ + \theta_+\theta_- F_{+-},$$

and

$$D_+ = \frac{\partial}{\partial\theta^+} + \theta^+\frac{\partial}{\partial\sigma^-}, \qquad D_- = \frac{\partial}{\partial\theta^-} + \theta^-\frac{\partial}{\partial\sigma^+}$$

are (1,1) superspace covariant derivatives. Please note that the $\pm$ indices on the θs are 2d spin, *i.e.* sign of charge under the 2d $SO(2) \sim U(1)$ of rotations; so for example the object $D_\pm$ has spin $\pm 1/2$. The conservation of this quantity is a useful check on the calculation.

3. **$\mathcal{N} = (0,2)$ and $\mathcal{N} = (2,2)$ supersymmetry.**

In the previous problem we studied a system with $(1,1)$ supersymmetry. Many interesting theories have extended supersymmetry in two dimensions. A particularly interesting and familiar case is $2d, \mathcal{N} = (2,2)$ supersymmetry, which arises by dimensional reduction from (the conceivably realistic) $4d, \mathcal{N} = 1$ supersymmetry. [2]

[2]For this problem, we return to a euclidean worldsheet.

(a) Consider the action

$$\int d^2z \int d^2\theta \ \bar{D}\mathbf{X}D\mathbf{X}$$

where now θ is a *complex* grassmann variable, $d^2\theta = d\theta d\bar{\theta}$,

$$\mathbf{X}(z, \bar{z}, \theta, \bar{\theta}) \equiv X + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$$

are 2d $\mathcal{N} = 2$ superfields and

$$D = \frac{\partial}{\partial\theta} + \theta\partial_z, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \bar{\theta}\bar{\partial}_{\bar{z}}$$

are $\mathcal{N} = 2$ superspace covariant derivatives. What does this look like in components?

(b) Now we will discuss $\mathcal{N} = (2, 2)$ supersymmetry. This means that we have a complex grassmann superspace coordinate of both chiralities $\alpha = \pm\frac{1}{2}$: $\theta_+, \bar{\theta}_+, \theta_-, \bar{\theta}_-$. And therefore we have four superspace derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\rho^a_{\alpha\beta}\bar{\theta}^\beta\partial_a, \qquad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\rho^a_{\alpha\beta}\theta^\beta\partial_a.$$

A *chiral* superfield is one which is killed by half the supercharges:

$$\bar{D}_\pm\Phi = 0.$$

Such a field can be expanded as ($\alpha = \pm$)

$$\Phi(x, \theta, \bar{\theta}) = \phi(y) + \sqrt{2}\theta_\alpha\psi^\alpha(y) + \theta^\alpha\theta_\alpha F$$

where

$$y^a = x^a + i\theta^\alpha\sigma^a_{\alpha\beta}\bar{\theta}^\beta$$

and the $\pm$ indices are raised and lowered with $\epsilon^{\alpha\beta}$.

Show that the action

$$S_{\text{canonical}} = \int d^2z \int d^2\theta_+ d^2\theta_- \ \bar{\Phi}\Phi \quad .$$

gives a canonical kinetic term for X and its superpartner, but no term with derivatives of the *auxiliary field* F. [3]

[3] It will be useful to note that the complex conjugate field $\bar{\Phi}$ is an antichiral multiplet satisfying $\bar{D}_\alpha\bar{\Phi} = 0$, and can be expanded as

$$\bar{\Phi} = \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}_\alpha\bar{\psi}^\alpha(\bar{y}) + \bar{\theta}_\alpha\bar{\theta}^\alpha F$$

where $\bar{y}^a = x^a - i\theta^\alpha\sigma^a_{\alpha\beta}\bar{\theta}^\beta$.

A more general kinetic term comes from a *Kähler potential* K,

$$S_K \int d^2z \int d^2\theta_+ d^2\theta_- \; K(\bar{\Phi}, \Phi) \quad ,$$

where before we made the special choice $K = \bar{\Phi}\Phi$. What are the bosonic terms coming from this superspace integral?

Now consider a *superpotential* term, which can be written as an integral over only half of the superspace:

$$S_W = \int d^2z \int d\theta_+ d\theta_- W(\Phi) + \text{h.c.}.$$

Show that this term is supersymmetric if W depends only on chiral superfields in a holomorphic way, $\frac{\partial}{\partial \bar{\Phi}} W = 0$.

With the action

$$S = S_K + S_W$$

integrate out the auxiliary fields $F, \bar{F}$ to find the form of the bosonic potential for ϕ. [4] Describe the supersymmetric ground states of this system.

4. **Supersymmetric ghosts.** Consider a (chiral) $\mathcal{N} = 2$ multiplet of ghosts:

$$B = \beta + \theta b, \quad C = c + \theta\gamma \quad ;$$

here θ is a coordinate on 2d $\mathcal{N} = 2$ superspace, b, c are ordinary Grassmann bc ghosts, with scaling weight $\lambda, 1 - \lambda$. β, γ are *commuting* ghosts with weights $\lambda - \frac{1}{2}$ and $\frac{1}{2} - \lambda$ respectively.

Write the action

$$S_{BC} = \int d^2z \int d^2\theta \; B\bar{D}C$$

in components; here $\bar{D} = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta}\bar{\partial}_{\bar{z}}$ is the same superspace covariant derivative from before.

Find an expression for the supercurrent in this theory.

[4] It is often useful to label a superfield by its lowest component.