MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Problem Set 5

A little more on open strings, bosonization, superstring spectrum
Reading: Polchinski, Chapter 10.

Due: Thursday, November 8, 2007 at 11:00 AM in lecture.

1. **The open string tachyon is in the adjoint rep of the Chan-Paton gauge group.**

   Convince yourself that I wasn't lying when I said that the pole in the Veneziano amplitude (with no CP factors) at $s = 0$ cancels in the sum over orderings. Convince yourself that this means that when CP factors are included the tachyon is in the adjoint representation of the D-brane worldvolume gauge group.

2. **Bosonization of a Dirac fermion = Fermionization of a non-chiral boson.**

   (a) Consider the CFT associated with compactification on a single circle of radius R, *i.e.* one periodic free boson $X \simeq X + 2\pi R$. Show that the partition function on a torus of modular parameter $q = e^{2\pi i\tau}$ is (in $\alpha' = 2$ units)

$$Z_R(\tau, \bar\tau) = \text{tr } q^{L_0 - \frac{1}{24}} \bar{q}^{\tilde{L}_0 - \frac{1}{24}}$$
$$= \frac{1}{|\eta|^2} \sum_{n,m\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{n}{R} + \frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R} - \frac{mR}{2}\right)^2} \quad ,$$

   where the Dedekind eta function is

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

   Note that this function is invariant under T-duality:

$$Z_R = Z_{\alpha/R}.$$

   (b) Here we will study the special radius $R = 1 = \sqrt{\alpha'/2}$ (or equivalently $R = 2 = \sqrt{2\alpha'}$, by T-duality). Show that at this special radius (which is different from the self-dual radius, $R = \sqrt{2} = \sqrt{\alpha'}$!), the partition function can be written as

$$Z_1(\tau, \bar{\tau}) = \frac{1}{2} \frac{1}{|\eta|^2} \left(\left| \sum_n q^{n^2/2} \right|^2 + \left| \sum_n (-1)^n q^{n^2/2} \right|^2 + \left| \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \right|^2 \right) .$$

(c) Show that this last form of Z is the partition function of a 2d *Dirac* fermion (!). Note that 'Dirac fermion' here means two left-moving MW fermions and two right-moving MW fermions, and we are choosing the spin structures of the right-moving and left-moving fermions in a correlated, non-chiral way – the GSO operator is the $(-1)^F$ which counts the fermion number of all the fermions at once, and we include only RR and NSNS sectors. This is called the 'diagonal modular invariant'. Note that this is a different sum over spin structures than the one in the system bosonized in Polchinski chapter 10 (and this is why it can be modular invariant with fewer than eight fermions).

[**Hint:** (i) The three terms in Z_1 arise from the three choices of spin structure which give nonzero partition functions.

(ii) The sums in the squares are theta functions, specifically,

$$\theta_3(\tau) = \vartheta_{00}(0|\tau) = \sum_n q^{n^2/2}$$

$$\theta_4(\tau) = \vartheta_{01}(0|\tau) = \sum_n (-1)^n q^{n^2/2}$$

$$\theta_2(\tau) = \vartheta_{10}(0|\tau) = \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \quad ,$$

which can be expressed as infinite products (instead of infinite sums), as described on page 215 of Polchinski vol. I. Rewrite $Z_1(\tau, \bar{\tau})$ using the product forms of the theta functions.]

3. **Superstring worldsheet vacuum energy.**

   Show that

$$\sum_{n=0}^{\infty} (n - j) - \sum_{n=0}^{\infty} n = -\frac{1}{2} j(j-1),$$

where we can define the divergent sums by a regulator mass:

$$\sum_{n=0}^{\infty} \omega_n \equiv \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \omega_n e^{-\epsilon \omega_n} \quad .$$

Show that this reproduces the lightcone gauge vacuum energies for the NS and R sectors.

Relatedly, you might want to do Polchinski problem 10.8.

4. **bispinors.**

   Make yourself happy about the field content of the RR sectors of the type II superstrings. In particular, if $\eta_\pm$ are chiral spinors,

$$(1 \mp \gamma)\eta_\pm = 0, \qquad \{\gamma, \gamma^i\} = 0, \forall i = 1..8,$$

   show that

$$\tilde{\eta}_+ \gamma^{i_1 \dots i_q} \eta_+ = 0$$

   if q is odd and

$$\tilde{\eta}_+ \gamma^{i_1 \dots i_q} \eta_- = 0$$

   if q is even.