MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2007

Solution Set 5

A little more on open strings, bosonization, superstring spectrum

Reading: Polchinski, Chapter 10.

Due: Thursday, November 8, 2007 at 11:00 AM in lecture.

1. **The open string tachyon is in the adjoint rep of the Chan-Paton gauge group.**

   Convince yourself that I wasn't lying when I said that the pole in the Veneziano amplitude (with no CP factors) at $s = 0$ cancels in the sum over orderings. Convince yourself that this means that when CP factors are included the tachyon is in the adjoint representation of the D-brane worldvolume gauge group.

   The Veneziano amplitude is

$$S_{D_2}(k_1..k_4) = 2ig_0^4 C_{D^2}\tilde{\delta}(\sum k)\left(I(s,t) + I(t,u) + I(u,s)\right)$$

   with

$$I(s,t) = \int_0^1 dy y^{-\alpha' s-2}(1-y)^{-\alpha' t-2},$$

   and $\alpha'(s+t+u) = -4$. **The** 2 **out front was the sum over orderings of 2 and 3, or alternatively the sum over the two orientations of the boundary. An analytic continuation of this which allows us to study the region near** $s = 0$ **(the integral representation doesn't converge there) is**

$$I(s,t) = \frac{\Gamma(-\alpha' s-1)\Gamma(-\alpha' t-1)}{\Gamma(-\alpha' s-\alpha' t-2)}.$$

   Near $s \to 0$**, this has a pole of the form:**

$$I(s \to 0, t) \sim \frac{\alpha t + 2}{\alpha' s}.$$

   The t-channel diagram gives

$$I(u, s \to 0) \sim \frac{\alpha' u + 2}{\alpha' s},$$

while the other channel $I(t,u)$ is regular when $t+u \sim -4$. So the total residue is

$$\frac{1}{\alpha'}\left(\alpha' t + 2 + \alpha' u + 2\right)|_{s=0} = 0.$$

With CP factors, the amplitude is instead

$$S_{D_2}(k_1, \lambda_1; ..k_4, \lambda_4) = i g_0^4 C_{D^2} \tilde{\delta}(\sum k) \times$$

$$[I(s,t)\mathrm{tr}\,(1234 + 4321) + I(t,u)\mathrm{tr}\,(4231 + 1324) + I(u,s)\mathrm{tr}\,(1243 + 3421)]$$

where

$$1234 \equiv \lambda_1 \lambda_2 \lambda_3 \lambda_4$$

and we determined the order of the CP matrices by the relative orderings of the vertex operators in the $y = y_4$ integral; note that the two orientations of the boundary are no longer the same. The two terms (first and third) that contribute a $s = 0$ pole are related by $4 \leftrightarrow 3$ and so the sum of residues is now proportional to

$$\alpha'^{-1}\left((\alpha' t + 2)\,\mathrm{tr}\,(1234 + 4321) + (\alpha' u + 2)\,\mathrm{tr}\,(1243 + 3421)\right) = (t-u)\mathrm{tr}[\lambda_1, \lambda_2][\lambda_3, \lambda_4].$$

Unitarity then relates this residue to the three-point coupling to the gauge boson:

$$\mathcal{A}_{D^2}(k_1..k_4) = i \int \frac{d^{26}k}{(2\pi)^{26}} \sum_{\zeta} \frac{\mathcal{A}_{D^2}(k_1, k_2; k, \zeta)\mathcal{A}_{D^2}(-k, \zeta; k_3, k_4)}{-k^2 + i\epsilon} + \text{reg. at } s = 0.$$

If the tachyons are in the adjoint, the three-point coupling between the gauge boson and the two tachyons looks like

$$\mathrm{tr}(DT)^2 \equiv \mathrm{tr}(\partial T - [A,T])^2 \mathrm{tr}\partial T[A,T] \propto \mathrm{tr}\lambda_2[\lambda_A, \lambda_1] \propto f_{A12}$$

(f_{ABC} are the structure constants of the gauge group). Then the tree-level s-channel diagram has the group theory structure

$$\sum_A f_{A12} f_{A34} \sim \mathrm{tr}[\lambda_1, \lambda_2][\lambda_3, \lambda_4]$$

where A is the adjoint index of the gauge boson. This is exactly what we found. (And the momentum dependence $u - t$ comes from the derivative acting on the scalar.)

2. **Bosonization of a Dirac fermion = Fermionization of a non-chiral boson.**

(a) Consider the CFT associated with compactification on a single circle of radius R, *i.e.* one periodic free boson $X \simeq X + 2\pi R$. Show that the partition function on a torus of modular parameter $q = e^{2\pi i\tau}$ is (in $\alpha' = 2$ units)

$$Z_R(\tau, \bar{\tau}) = \mathrm{tr} q^{L_0 - \frac{1}{24}} \bar{q}^{\tilde{L}_0 - \frac{1}{24}}$$
$$= \frac{1}{|\eta|^2} \sum_{n,m\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2} \quad ,$$

where the Dedekind eta function is

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Note that this function is invariant under T-duality:

$$Z_R = Z_{\alpha/R}.$$

Our expression for L_0 in terms of oscillators gives

$$Z_R(\tau, \bar{\tau}) = \mathrm{tr} q^{-\frac{1}{24}} \bar{q}^{-\frac{1}{24}} \sum_{p_L, p_R} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \prod_n \sum_{N_n, \tilde{N}_n} q^{nN_n} \bar{q}^{n\tilde{N}_n}$$

where

$$p_L = \frac{n}{R} + \frac{mR}{2}, \quad p_R = \frac{n}{R} - \frac{mR}{2}, \quad n, m \in \mathbb{Z}$$

are the allowed momenta. The bosonic oscillator sums are geometric and using $\eta^{-1}(q) \equiv q^{-1/24} \prod_{n=1}^{\infty}(1-q^n)$ we have

$$Z_R = \frac{1}{|\eta|^2} \sum_{n,m\in\mathbf{Z}} q^{\frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2} \quad .$$

(b) Here we will study the special radius $R = 1 = \sqrt{\alpha'/2}$ (or equivalently $R = 2 = \sqrt{2\alpha'}$, by T-duality). Show that at this special radius (which is different from the self-dual radius, $R = \sqrt{2} = \sqrt{\alpha'}$!), the partition function can be written as

$$Z_1(\tau, \bar{\tau}) = \frac{1}{2} \frac{1}{|\eta|^2} \left(\left| \sum_n q^{n^2/2} \right|^2 + \left| \sum_n (-1)^n q^{n^2/2} \right|^2 + \left| \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \right|^2 \right).$$

The trick here is just to break up the momentum sums into integers and half-integers. (From our discussion of bosonization you know that this is a good idea because the integer momenta will be NS states ($\psi \sim e^{iH}$) and the half-integer momenta will be R states ($\Theta \sim e^{\frac{1}{2}iH}$).) The momentum sum is

$$|\eta|^2 Z_1 = \sum_{n,m\in\mathbf{Z}} q^{\frac{1}{2}\left(n+\frac{m}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(n-\frac{m}{2}\right)^2} = \sum_{n,r\in\mathbf{Z}} \left(q^{\frac{1}{2}(n+r)^2} \bar{q}^{\frac{1}{2}(n-r)^2} + q^{\frac{1}{2}\left(n+r+\frac{1}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(n-r-\frac{1}{2}\right)^2} \right)$$

In the first term, $r = 2m$; in the second $r = 2m+1$. These will be related to the R and NS sectors of the fermion, respectively. Now, we would like rewrite this as a sum over 'conformal blocks', *i.e.* as a sum of products $\sum_i M_{ij} f_i(q) \tilde{f} j(\bar{q})$; M will turn out to be diagonal. To do this, define $a = n+r, b = n-r$. Notice that if n, m are integers then a, b always have the same parity. We can implement this constraint by inserting the projector $P = \frac{1}{2}\left(1+(-1)^{a+b}\right)$: for any f

$$\sum_{n,m\in\mathbf{Z}} f(n+m, n-m) = \frac{1}{2} \sum_{a,b\in\mathbf{Z}} \left(1+(-1)^{a+b}\right) f(a,b) \quad ;$$

this is the (diagonal) GSO projection on fermion number. We find

$$|\eta|^2 Z_1 = \sum_a q^{\frac{1}{2}a^2} \sum_b \bar{q}^{\frac{1}{2}b^2} + \sum_a (-1)^a q^{\frac{1}{2}a^2} \sum_b (-1)^b \bar{q}^{\frac{1}{2}b^2} +$$

$$\sum_a q^{\frac{1}{2}\left(a+\frac{1}{2}\right)^2} \sum_b \bar{q}^{\frac{1}{2}\left(b-\frac{1}{2}\right)^2} + \sum_a (-1)^a q^{\frac{1}{2}\left(a+\frac{1}{2}\right)^2} \sum_b (-1)^b \bar{q}^{\frac{1}{2}\left(b-\frac{1}{2}\right)^2}$$

The last term on the RHS vanishes since the summand is odd under $a \to -a$ (and $b \to -b$, too, so it's twice as zero). We're left with

$$|\eta|^2 Z_1 = \left| \sum_n q^{n^2/2} \right|^2 + \left| \sum_n (-1)^n q^{n^2/2} \right|^2 + \left| \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \right|^2$$

The sums in the squares are theta functions, specifically,

$$\theta_3(\tau) = \vartheta_{00}(0|\tau) = \sum_n q^{n^2/2}$$

$$\theta_4(\tau) = \vartheta_{01}(0|\tau) = \sum_n (-1)^n q^{n^2/2}$$

$$\theta_2(\tau) = \vartheta_{10}(0|\tau) = \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \quad ,$$

which can be expressed as infinite products (instead of infinite sums), as described on page 215 of Polchinski vol. I. Rewriting $Z_1(\tau, \bar{\tau})$ using the product forms of the theta functions I get

$$Z_1 = \frac{1}{2}\left|q^{-\frac{1}{24}}\right|^2 \left(\left|\prod_{r=1}^{\infty}(1+q^{r-\frac{1}{2}})^2\right|^2 + \left|\prod_{r=1}^{\infty}(1-q^{r-\frac{1}{2}})^2\right|^2 + \left|2q^{\frac{1}{8}}\prod_{r=1}^{\infty}(1+q^r)^2\right|^2 \right)$$

(c) Show that this last form of Z is the partition function of a 2d *Dirac* fermion (!). Note that 'Dirac fermion' here means two left-moving MW fermions and two right-moving MW fermions, and we are choosing the spin structures of the right-moving and left-moving fermions in a correlated, non-chiral way – the GSO operator is the $(-1)^F$ which counts the fermion number of all the fermions at once, and we include only RR and NSNS sectors. This is called the 'diagonal modular invariant'. Note that this is a different sum over spin structures than the one in the system bosonized in Polchinski chapter 10 (and this is why it can be modular invariant with fewer than eight fermions).

[**Hint:** (i) The three terms in Z_1 arise from the three choices of spin structure which give nonzero partition functions.

(ii) The sums in the squares are theta functions, specifically,

$$\theta_3(\tau) = \vartheta_{00}(0|\tau) = \sum_n q^{n^2/2}$$

$$\theta_4(\tau) = \vartheta_{01}(0|\tau) = \sum_n (-1)^n q^{n^2/2}$$

$$\theta_2(\tau) = \vartheta_{10}(0|\tau) = \sum_n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \quad ,$$

which can be expressed as infinite products (instead of infinite sums), as described on page 215 of Polchinski vol. I. Rewrite $Z_1(\tau, \bar{\tau})$ using the product forms of the theta functions.]

In the NS sector, the fermions are half-integer moded (this is really the NSNS sector, *i.e.* NS on both sides).

$$Z_{NS} = \mathrm{tr}_N S e^{-\tau_2 H + i\tau_2 P}\frac{1}{2}(1 + (-1)^{F+\tilde{F}}).$$

Remembering that each mode can be occupied at most once,

$$Z_{NS} = \frac{1}{2}\left(\left|q^{E_0^{NS}}\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})\right|^2 \left|q^{E_0^{NS}}\prod_{m=1}^{\infty}(1-q^{m-\frac{1}{2}})\right|^2\right).$$

Using the zeropoint energy mnemonic, $E_0^{NS} = 2(-\frac{1}{48})$ **for two antiperiodic fermions, and we get:**

$$Z_{NS} = \frac{1}{2}\left(\left|\frac{\theta_{00}}{\eta}\right|^2 + \left|\frac{\theta_{01}}{\eta}\right|^2\right).$$

In the R sector, the fermions are integer-moded, including zero, so there are four degenerate groundstates from $\{\psi_0, \psi_0^\star\} = 1$ **and** $\tilde{\psi}_0, \tilde{\psi}_0^\star = 1$**. These groundstates have opposite fermion numbers in pairs, hence**

$$Z_R = \mathrm{tr}_R e^{-\tau_2 H + i\tau_2 P}\frac{1}{2}(1+(-1)^{F+\tilde{F}}).$$

$$= \frac{1}{2}\left(\left|q^{E_0^R}(1+1)\prod_{m=1}^{\infty}(1+q^m)\right|^2 + \left|q^{E_0^R}(1-1)\prod_{m=1}^{\infty}(1-q^m)\right|^2\right).$$

Using the zeropoint energy for two periodic fermions, $E_0^R = 2\frac{1}{24}$**, we get directly the product version of the theta functions**

$$Z_R = \frac{1}{2}\left|\frac{\theta_{10}}{\eta}\right|^2$$

and altogether we've reproduced

$$Z_1 = Z_{NS} + Z_R.$$

3. **Superstring worldsheet vacuum energy.**

   Show that[1]

$$\sum_{n=0}^{\infty}(n-j) - \sum_{n=0}^{\infty} n = -\frac{1}{2}j(j+1),$$

[1] Sorry for the typo here in the statement of the problem.

where we can define the divergent sums by a regulator mass:

$$\sum_{n=0}^{\infty} \omega_n \equiv \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} \omega_n e^{-\epsilon \omega_n} \quad .$$

Show that this reproduces the lightcone gauge vacuum energies for the NS and R sectors.

Relatedly, you might want to do Polchinski problem 10.8.

$$Z_\epsilon(j) \equiv \sum_{n=0}^{\infty} (n-j) e^{-\epsilon(n-j)} = -\partial_\epsilon \sum_{n=0}^{\infty} e^{-\epsilon(n-j)} = -\partial_\epsilon \left(\frac{e^{\epsilon j}}{1-e^{-\epsilon}} \right)$$

$$= e^{\epsilon j} \frac{j(1-e^{-\epsilon}) - (e^{-\epsilon})}{(1-e^{-\epsilon})^2} = e^{\epsilon j} \frac{j - (j+1)e^{-\epsilon}}{(1-e^{-\epsilon})^2}$$

It's not an accident that this looks like the generating function of Bernoulli numbers. So

$$\sum_n n e^{-n\epsilon} = Z_\epsilon(j=0) = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2}$$

and the vacuum energy is the small ϵ limit of

$$Z_\epsilon(j) - Z_\epsilon(0) = \frac{(j+1)e^{\epsilon(j-1)} - je^{+\epsilon j} - e^{-\epsilon}}{(1-e^{-\epsilon})^2} = \frac{\frac{1}{2}(j+1)(j-1)^2 - \frac{1}{2}j^3 - \frac{1}{2} + \mathcal{O}(\epsilon^3)}{\epsilon^2 + \mathcal{O}(\epsilon^3)}.$$

Notice that the singular $\epsilon^{-2}, \epsilon^{-1}$ terms cancel between the bose and fermi contributions. This is

$$Z_\epsilon(j) - Z_\epsilon(0) = \frac{-\frac{1}{2}j(j+1)\epsilon^2 + \mathcal{O}(\epsilon^3)}{\epsilon^2 + \mathcal{O}(\epsilon^3)}$$

which gives

$$E_0 = \lim_{\epsilon \to 0} \left(Z_\epsilon(j) - Z_\epsilon(0)\right) = -\frac{1}{2} j(j+1).$$

For the NS sector of the lightcone superstring, 4 complex periodic bosons and 4 complex antiperiodic fermions give

$$-4 \times (-\frac{1}{2} j(j+1)) = 4\frac{1}{2}(-\frac{1}{2})(1-\frac{1}{2}) = -\frac{1}{2}.$$

For the R sector, the change of fermion periodicity gives

$$E_0^R = -4 \times (-\frac{1}{2} \times 0) = 0,$$

as required by supersymmetry.

For the NS sector, we find the contribution of one *real* antiperiodic ($j = -\frac{1}{2}$) fermion to be

$$E_0^{NS}/\text{per fermion} = -\frac{1}{2}\sum_{n=0}^{\infty}(n+\frac{1}{2}) = \frac{1}{24} - \frac{1}{2}(-\frac{1}{2})(1-\frac{1}{2}) = \frac{1}{24} - \frac{1}{16} = -\frac{1}{48}.$$

For the R sector, we find the contribution of one *real* periodic ($j = 0$) fermion to be

$$E_0^R/\text{per fermion} = -\frac{1}{2}\sum_{n=0}^{\infty} n = \frac{1}{24} - \frac{1}{2}(0)1 = \frac{1}{24}.$$

4. **bispinors.**

Make yourself happy about the field content of the RR sectors of the type II superstrings. In particular, if $\eta_\pm$ are chiral spinors,

$$(1 \mp \gamma)\eta_\pm = 0, \quad \{\gamma, \gamma^i\} = 0, \forall i = 1..8,$$

show that

$$\tilde{\eta}_+ \gamma^{i_1 \dots i_q} \eta_+ = 0$$

if q is odd and

$$\tilde{\eta}_+ \gamma^{i_1 \dots i_q} \eta_- = 0$$

if q is even.

There are two basic ideas. The first is that in tensoring together the two spinors, we need only stick antisymmetrized combinations of γs. This is true because the gammas satisfy $\{\gamma^i, \gamma^j\} = 2\eta^{ij}$ which means that any symmetric part can be reduced to a lower-rank antisymmetric part.

The second point is that on a chiral spinor, one can multiply for free by the chirality projector:

$$\eta_\pm = \frac{1}{2}(1 \pm \gamma)\eta_\pm.$$

And since moving the γ through a γ^i gives a minus ($\{\gamma, \gamma^i\} = 0$), we have

$$\tilde{\eta}_+ \gamma^{i_1 \ldots i_q} \eta_\pm = \tilde{\eta}_+ \gamma^{i_1} \ldots \gamma^{i_q} \frac{1}{2}(1 \pm \gamma)\eta_\pm \quad \pm \quad \text{perms}$$

$$= \tilde{\eta}_+ \gamma^{i_1} \ldots \gamma^{i_{q-1}} \frac{1}{2}(1 \mp \gamma)\gamma^{i_q} \eta_\pm \quad \pm \quad \text{perms}$$

$$= \tilde{\eta}_+ \frac{1}{2}(1 + (\pm)^q \gamma)\gamma^{i_1} \ldots \gamma^{i_q} \frac{1}{2}(1 \pm \gamma)\eta_\pm \quad \pm \quad \text{perms}$$

which is zero if

$$(\pm)^q = -1$$

which is when q is odd for $+$ and when q is even for $-$.