8.821 F2008 Lecture 05

Lecturer: McGreevy   Scribe: Evangelos Sfakianakis

September 22, 2008

Today

1. Finish hindsight derivation

2. What holds up the throat?

3. Initial checks (counting of states)

4. next time: Supersymmetry self-defense

Previously, we had three hints for interpreting the bold Assertion that we started with, that hidden inside any non-Abelian gauge theory is a quantum theory of gravity.

1. The Weinberg-Witten theorem suggests that the graviton lives on a different space than the Quantum Field Theory.

2. This comes from the holographic principle. The idea, motivated by black hole thermodynamics, is that the theory of gravity should have a number of degrees of freedom that grows more slowly than the volume (non-extensive) suggesting that the quantum gravity should live is some extra (more) dimensions than the QFT.

3. The structure of the Renormalization Group suggests that we can identify one of these extra dimensions as the RG-scale.

In order to make these statements a little more concrete, we will make a number of simplifying assumptions. That means we will not start by writing down the more general case of the correspondence, but we will start with a special case.

1. There are many colors in our non-Abelian gauge theory. The motivation for this is best given by the quote:

   "You can hide a lot in a large-N matrix."

The idea is that at large N the QFT has many degrees of freedom, thus corresponding to the limit where the extra dimensions will be macroscopic.

2. We will work in the limit of strong coupling. The motivation for that lies in the fact that we know some things about weakly coupled Field Theories, and they don't seem like Quantum Gravity. This constrains the manner in which we take N to be large, since we would still like the system to be interacting. This means that we don't want vector models.

We want the theory to have as much symmetry as possible. However there is a theorem that stands in the way of this.

Theorem (Coleman-Mandula) : All bosonic symmetries of a "sensible" S-matrix (nontrivial, having finite matrix elements) belong to the Poincare group.

We are going to use all possible loopholes to circumvent this theorem. The first lies in the word "bosonic" and the second one in "sensible".

3. SuperSymmetry (SUSY)

   (a) It constrains the form of the interactions. This means that there are fewer candidates for the dual.

   (b) Supersymmetric theories have more adiabatic invariants, meaning observables that are independent of the coupling. So there are more ways to check the duality.

   (c) It controls the strong-coupling behavior. The argument is that in non-SUSY theories, if one takes the strong-coupling limit, it tends not to exist. The examples include QED (where one hits the landau Pole for strong couplings) and the Thirring model, which is exactly solvable, yet doesn't exist above some coupling.

   (d) Nima says it's simpler (`http://arxiv.org/abs/0808.1446`).

   The most supersymmetric theory in d=4 is the $\mathcal{N}=4$ SYM and we'll be focusing on this theory for a little while. It has the strange property that, unlike other gauge theories, the beta function of the gauge coupling is exactly zero. This means that the gauge coupling is really a dimensionless parameter, and so, reason number (e) to like SUSY is...

   (e) SUSY allows a line of fixed points. This means that there is a dimensionless parameter which interpolates between weak and strong coupling.

4. Conformal Invariance: The fact the coupling constant is dimensionless (even quantum-mechanically) says that there is scale invariance. The S-matrix is not finite! Everything is soft gluons.

Scale invariance applies to both space and time, so $X^\mu \to \lambda X^\mu$,where $\mu = 0, 1, 2, 3$. As we said, the extra dimension coordinate is to be thought of as an energy scale. Dimensional analysis suggests that this will scale under the scale transformation, so $z \to \frac{z}{\lambda}$. The most general five-dimensional metric (one extra dimension) with this symmetry and Poincare invariance is of the following form:

$ds^2 = (\frac{\tilde{z}}{\tilde{L}})^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\tilde{z}^2}{\tilde{z}^2} L^2$. Let $\tilde{z} = \frac{\tilde{L}}{L}\tilde{z} \Rightarrow \tilde{L} = L$ We can now bring it into the more familiar form (by change of coordinates) $ds^2 = (\frac{z}{L})^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dz^2}{z^2} L^2$, which is AdS_5. It turns out that this metric also has conformal invariance. So scale and Poincaré implies conformal invariance, at least when there is a gravity dual. This is believed to be true more

generally, but there is no proof. Without Poincaré invariance, scale invariance definitely does *not* imply conformal invariance.

We now formulate the guess that a 4-dimensional Conformal field Theory is related to a theory of Gravity on AdS_5

A specific example is N=4 SYM which is related to IIB Supergravity on $AdS_5 \times S^5$.

Note: On a theory of gravity space-time is a dynamical variable, where you specify asymptotics. This CFT defines a theory of gravity on spaces that are asymptotically AdS_5.

Why is AdS_5 a solution?

1. Check on PSet

2. Use effective field theory (which in this case just means dimensional reduction) to elaborate on my previous statement that the reason why this is a solution is that "the flux holds it up". To do this let's consider the relevant part of the supergravity action, which is $S = \int d^D x [\sqrt{G} \mathcal{R}^{(D)} - F_g \wedge \star F_g]$
As an example consider the case where D=10 and g=5 (note that we are working in dimensions where $G_N^{(D)} = 1 = 1/5! = 2 = 4\pi$).
[Note: The reference for this is Denef et al hep-th/0701050]

Let's make an Ansatz (called Freund-Rubin) The metric is some space-time (x) plus a q-sphere of radius R, so that $ds^2 = g_{\mu\nu} dx^\mu dx^\nu + R(x)^2 G_{mn} dy^m dy^n$ and furthermore for the flux though the q-sphere we have $\int_{S^q} F_q = N$

To find the form of this part of the metric as a result of the flux, we integrate over the q-sphere and get the action in D-dimensions, which will contain a term which is

$\int_{S^q} \sqrt{G} G^{m_1 n_1}...G^{m_q n_q} (F_g)_{m_1...m_q} (F_g)_{n_1...n_q}$

We should then think of R(x) as a "moduli field". Now we have to evaluate the following integral $\int_{S^q} \sqrt{G} \mathcal{R}^{(D)} = \sqrt{g} R^q [\mathcal{R} + \frac{a}{R^2} + ...]$

We will now go to something called the FRAME GAME
To make the D-dimensional Einstein-Hilbert term canonical, we can do a Weyl-rescaling of the metric, meaning to define a new metric $g_{\mu\nu}^E = \lambda(x) g_{\mu\nu}$. So the new metric is some function of the space-time variables times the old metric.
So $\sqrt{|g^E|} = \lambda^{d/2} \sqrt{g}$, $\mathcal{R}^E = \lambda^{-1} \mathcal{R}$.
We'll pick λ to absorb the factor of R, so $\sqrt{g^E} \mathcal{R}^E = \lambda^{d/2-1} \sqrt{g} \mathcal{R}^{(d)} = R^q \sqrt{g} \mathcal{R}^{(d)}$, $\lambda = R^{2q/d-2}$.
We know that the flux is the integral over the q-sphere, so $\int_{S^q} F = N \Rightarrow N \sim \frac{N}{R^q}$, so the flux term of the potential is $V_{flux}(R) \equiv \int_{S^q} F \wedge \star F = Vol(S^q) \sqrt{g} F^2 = R^g \lambda^{-d/2} \sqrt{g^E} (\frac{n}{R^q})^2$

So $V_{flux}(R) \sim \frac{N^2}{R^\alpha} - \frac{1}{R^\beta}$. where $\alpha > \beta > 0$

At small R the first term in the effective potential dominates, while for R large, the effective potential approaches zero from below. In between there is some minimum. To find the actual value of this we take $0 = V'(R_{min}) \Rightarrow N^2 \sim R_{min}^{\alpha-\beta}$.

The fact that the potential goes to zero at large R is unavoidable. It is because when R is large, the curvature is small and flux is dilute. So all terms in the potential go away.
For special values of α and β, we get that $N \sim R_{min}^4 = L^4$

This endeavor is called "flux compactification" and it is the way to find vacua of string theory without massless scalars.

Furthermore, we can add some effective action for scalars with $m_R^2 \sim V''(R_{min})$, which looks like this : $S = \frac{1}{2} \int d^d x \sqrt{g}^e (R^E - 2\Lambda)^R$. The cosmological constant is determined as $\Lambda = V_{eff}(R_{min}) < 0$

The fact the the cosmological constant exists, means that the metric is not flat.

The equation of motion is the following: $0 = \frac{\partial S}{\partial g_{\mu\nu}} \Rightarrow R_{\mu\nu} = g_{\mu\nu}(\frac{1}{2}R - \Lambda)$

$\Rightarrow R = \frac{2d}{d-2}\Lambda \Rightarrow R_{\mu\nu}(\frac{-2\Lambda}{2-d})$

We know the solution for AdS_d, which has the metric $ds^2 = L^2 \frac{dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{r^2}$

Now I will describe the right way to compute curvatures, which is to use what is called "tetrad" or "vielbein" or "Cartan-Weyl" method.

There are three steps

Rewrite the metric as $ds^2 = (e^{\hat{r}})^2 + e^{\hat{\mu}}e^{\hat{\nu}}\eta_{\mu\nu}$,where I define $e^{\hat{r}} \equiv L\frac{dr}{r}$ and $e^{\hat{x}^\mu} \equiv L\frac{dx^\mu}{r}$

<u>Cartan 1</u> demands that they are covariantly constant and is formulated as follows: $de^a = -\omega^{ab}e^b$, where the ω is called the spin connection.

For example $de^{\hat{r}} = 0$ and $e^{\hat{\mu}} = -\frac{L}{r^2}dr \wedge dx^\mu = -\frac{1}{L}e^{\hat{r}} \wedge e^{\hat{\mu}} \Rightarrow \omega^{\hat{\mu}\hat{\nu}} = -\frac{1}{L}e^{\hat{\mu}} = -\omega^{\hat{r}\hat{\mu}}$

Why do we care about ω? Here's why:

<u>Cartan 2</u> $R^{ab} = d\omega^{ab} + \omega^{ac}\omega^{cb} = R^{ab}_{\mu\nu}dx^\mu \wedge dx^\nu$

From that you can get the Ricci tensor, which appears in the equation of motion, by contracting some indices $R^\mu_\nu = R^{\rho\mu}_{\rho\nu}$

Let's do some examples:
$R^{\hat{\mu}\hat{\nu}} = \omega^{\hat{\mu}\hat{r}}\omega^{\hat{r}\hat{\nu}} = -\frac{1}{L^2}e^{\hat{\mu}} \wedge e^{\hat{\nu}}$

$R^{\hat{\mu}\hat{r}} = d\omega^{\hat{\mu}\hat{r}} = \frac{1}{L^2}e^{\hat{r}} \wedge e^{\hat{\mu}}$

$R^{\mu\nu}_{\rho\sigma} = -\frac{1}{L^2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$

$R^{\mu r}_{\rho r} = -\frac{1}{L^2}\delta^\mu_\nu$

$R_{\mu\nu} = \frac{-d}{L^2}g_{\mu\nu} \Rightarrow \frac{2\Lambda}{2-d} = \frac{d}{L^2} \Rightarrow \Lambda = \frac{-d(d+1)}{2L^2}$

<u>Counting of degrees of freedom</u> [Susskind-Witten, hep-th/9805114]

The holographic principle tells us that the area of the boundary in planck units is the number of degrees of freedom (maximum entropy).

$\frac{\text{Area of boundary}}{4G_N} \overset{?}{=} \text{number of dof's of QFT} \equiv N_d$

Both sides are equal to infinity, so we need to regulate our counting somehow.

The picture we have of AdS is a collection of copies of Minkowski space of varying size and the boundary is the locus where they get really big. The area is defined as $A = \int_{R^3} \sqrt{g} d^3x = \int_{R^3} d^3x \frac{L^3}{r^3}$

This is infinite for two reasons: from the integral over x and from the fact that r is going to zero.

Let's regulate the Field Theory first. We put the thing on a lattice, introducing a low distance cut-off δ and we put it also in a box of size R. The number of degrees of freedom is the number of cells times N^2, so $N_d = \frac{R^3}{\delta^3} N^2$. To regulate the integral we write before, we stop not at $r = 0$ but we cut it off at $r = \delta$. $A_\delta = \int \int \int d^3x \frac{L^3}{r^3}|_{r=\delta} = \frac{R^3 L^3}{\delta^3}$

Now we can translate this

$\frac{L^2}{G_N^{10}} \sim N^2 \Rightarrow G_N^{(5)} = \frac{G_N^{(10)}}{L^5}$

$N_d = \frac{L^3 R^3}{G_N^{(5)} \delta^3} = \frac{A}{G_N^{(5)}}$

So AdS/CFT is indeed an implementation of the holographic principle.