8.821 F2008 Lecture 5: SUSY Self-Defense

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September 22, 2008

Today's lecture will teach you enough supersymmetry to defend yourself against a hostile supersymmetric field theory, should you meet one down a dark alleyway. Topics will include

1. SUSY representation theory, including basic ideas regarding the algebra and an explanation of why $\mathcal{N} = 4$ is maximal (a proof due to Nahm)
2. Properties of $\mathcal{N} = 4$ Super Yang-Mills theory, such as the spectrum and where it comes from in string theory.

Before plunging in, we might be inclined to wonder: why is supersymmetry so wonderful? In addition to the delightful properties that will be explored in the remainder of this course, there are various reasons arising from pure particle physics: for example, it stabilizes the electroweak hierarchy, changes the trajectory of RG flows such that the gauge couplings unify at very high energies, and halves the exponent in the cosmological constant problem. These are issues that we will not explore further in this lecture.

1 SUSY Representation Theory in $d = 4$

We begin with a quick review of old-fashioned Poincare symmetry in $d = 4$:

1.1 Poincare group

Recall that the isometry group of Minkowski space is the Poincare group, consisting of translations, rotations, and boosts. The various charges associated with the subgroups of the Poincare group are quite familiar and are listed below

	Charge	**Subgroup**
Translations	P_μ	$\mathbb{R}^{3,1}$
Rotations/boosts	$M_{\mu\nu}$	SO(3,1) $\sim$ (SU(2) $\times$ SU(2))

Luckily the rotation/boost part of the algebra SO(3,1) decomposes (at least in the sense of the "sloppy representation theory" that we are doing today) into two copies of SU(2). We are all familiar with representations of SU(2), each of which are labeled by a half-integer $s_i \in 0, 1/2, 1, ...$; the product of two of these reps gives us a rep of SO(3,1):

	(s_1, s_2)
Scalar	$(0,0)$
4-vector	$(1/2,1/2)$
Symmetric tensor	$(1,1)$
Weyl fermions	$(1/2,0)$, $(0,1/2)$
Self-dual,anti-self-dual tensors	$(1,0)$, $(0,1)$

Having mastered this group, we wonder whether we could possibly have a larger spacetime symmetry group to deal with, leading us to the idea of *enlarging the Poincare group by adding fermionic generators!*

1.2 Supercharges

Let's call these fermionic generators Q^I_α; α is a two-compnent Weyl spinor index belonging to the $(1/2,0)$ spinor representation above; essentially this simply defines the commutation relations of Q with $M_{\mu\nu}$, telling us that it transforms like a spinor. I labels the number of supersymmetries and thus goes from 1 to $\mathcal{N}$. The total number of supercharges is thus equal to the number $\mathcal{N}$ multiplied by the dimension of the spinor representation and so in four dimensions is $4\mathcal{N}$.

The adjoint of this operator is called $\bar{Q}_{\dot{\alpha}I} \equiv (Q^I_\alpha)^\dagger$. We note that $[P_\mu, Q] = 0$ and the anticommutators of Q with itself are given by

$$\left\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_k \qquad \{Q^I_\alpha, Q^J_\beta\} = 2\epsilon_{\alpha\beta} Z^{IJ} \tag{1}$$

Here Z^{IJ} is antisymmetric in its indices and thus exists only for $\mathcal{N} > 1$; playing with the SUSY algebra assures us that it commutes with everything and so is just a number called the central charge.

Note that this algebra is preserved under the action of a $U(\mathcal{N})$ called (inexplicably) an R-symmetry.

$$Q^I_\alpha \to U^I_J Q^J_\alpha \qquad U^I_J \in U(\mathcal{N}) \tag{2}$$

Whether or not the diagonal phase rotation of this $U(\mathcal{N})$ survives in the full quantum theory depends on dertails of the theory. Finally, the current j_R of this symmetry does not commute with the supercharges $[j_R, Q] \neq 0$, meaning different elements of the same supermultiplet can have different R-charges.

1.3 Representations of the SUSY Algebra: Massless states

Next, let us construct representations of this algebra, starting with massless states. Consider the Hilbert space $\mathcal{H}$ of a single massless particle, and pick a frame where the momentum is $P_\mu = (E, 0, 0, E)$, $E > 0$. Now using (1) and standard formulas for $\sigma_{\alpha\dot{\beta}}$ ($\sigma^0 = \mathbb{I}_2$ and σ^i are the Pauli matrices) we see that

$$\left\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\right\} = \delta^I_J \begin{pmatrix} 4\mathrm{E} & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

Focusing on the bottom left corner of this matrix, we see that

$$\left\{Q^I_2, (Q^J_2)^\dagger\right\} = 0 \to ||Q^I_2|\psi\rangle||^2 = 0, \tag{4}$$

for all $|\psi\rangle \in \mathcal{H}$. If we assume this zero-norm state $Q^I_2|\psi\rangle$ is actually the zero state (which is the case in a unitary Hilbert space), then we conclude that half of the Q's annihilate every state in the space, implying also that $Z^{IJ} = 0$ in this subspace. We are left with the Q^I_1's; note each Q^I_1 lowers the helicity by 1/2 and each $\bar{Q}^I_1$ raises the helicity by 1/2. Finally, if we normalize the Q_1's appropriately we get

$$\left\{\frac{Q^I_1}{2\sqrt{E}}, \frac{\bar{Q}_{1J}}{2\sqrt{E}}\right\} = \delta^I_J \tag{5}$$

From now on we will suppress the 1 subscript. This is simply the algebra of a fermionic oscillator, which we know how to deal with: start with a "highest-helicity state" that is annihilated by $\bar{Q}^I|h\rangle = 0$ and hit it with Q^I's to build the Hilbert space:

$$|h\rangle, \quad Q^{I_1}|h\rangle, \quad Q^{I_1}Q^{I_2}|h\rangle, \quad \ldots \quad Q^{I_1}Q^{I_2}\ldots Q^{I_\mathcal{N}}|h\rangle \tag{6}$$

There are a total of $2^\mathcal{N}$ states because (as pointed out by T. Senthil), each operator Q^I may or may not be present in the string of Q's. Now we wonder: how big can $\mathcal{N}$ be? A theorem by Nahm answers this question. A useful lemma (related to the Weinberg-Witten theorem discussed previously) is:

A quantum field theory without gravity cannot contain massless states with helicity $|h| > 1$.

This means that we cannot apply Q too many times, as eventually we will end up with a state with helicity more negative than -1. This is a constraint on $\mathcal{N}$ and allows us to summarize all possibilities in the following (finite) table, in which each entry is the number of states with that given helicity in the corresponding supermultiplet:

Helicity	$\mathcal{N}=1$ gauge	$\mathcal{N}=1$ chiral	$\mathcal{N}=2$ gauge	$\mathcal{N}=2$ hyper	$\mathcal{N}=4$ gauge
1	1	0	1	0	1
$\frac{1}{2}$	1	1	2	(1+1)	4
0	-	(1+1)	(1+1)	(2+2)	6
$-\frac{1}{2}$	1	1	2	(1+1)	4
-1	1	0	1	0	1

Note that if $\mathcal{N} > 4$, then we will necessarily have states with helicity greater than 1. Thus the maximal supersymmetry for a nongravitational theory in $d = 4$ is $\mathcal{N} = 4$, corresponding to 16 supercharges. If gravity is allowed, the corresponding bound on helicity is $|h| < 2 \to \mathcal{N} < 8$, and thus we can have $\mathcal{N} = 8$ supergravity with 32 supercharges in four dimensions.

1.4 Representations of the SUSY Algebra II: Massive states

It is time to move on to massive states, which turn out to have larger representations. Following a procedure analogous to that above, we boost to the rest-frame of the particle where we have $P_\mu = (M, 0, 0, 0)$. In that case (1) becomes

$$\left\{Q^I_\alpha, (Q^J_\beta)^\dagger\right\} = 2M\delta^I_J\delta^\alpha_\beta \qquad \left\{Q^I_\alpha, Q^J_\beta\right\} = 2\epsilon_{\alpha\beta}Z^{IJ} \tag{7}$$

It is now necessary to introduce some unfortunate notation. We take a deep breath and block diagonalize Z^{IJ} so that it takes the form

$$Z^{IJ} = \left(\begin{array}{cc|cc|c} 0 & Z_1 & & & \\ -Z_1 & 0 & & & \\ \hline & & 0 & Z_2 & \\ & & -Z_2 & 0 & \\ \hline & & & & \ldots \end{array}\right) \tag{8}$$

To keep track of this structure we split the index I into two parts $I = (\hat{I}, \bar{I})$, where $\bar{I}$ runs over each of the subblocks in the large matrix $\bar{I} = 1, 2, \ldots \mathrm{Floor}(\mathcal{N}/2)$ and $\hat{I} = 1, 2$ and runs over the two entries in each subblock. We now form the following linear combinations of $Q's$:

$$\mathcal{Q}^{\bar{I}}_{\alpha\pm} = \frac{1}{2}\left(Q^{(\hat{I}=1,\bar{I})}_\alpha \pm \sigma^0_{\alpha\dot{\beta}}\left(Q^{(\hat{I}=2,\bar{I})}_\beta\right)^\dagger\right) \tag{9}$$

These supercharges obey the anticommutation relations

$$\left\{\mathcal{Q}^I_{\alpha\pm}, \left(\mathcal{Q}^{\bar{J}}_{\beta\pm}\right)^\dagger\right\} = \delta^{\bar{I}}_{\bar{J}}\delta^\beta_\alpha(M \pm Z_{\bar{I}}), \tag{10}$$

where the $\pm$ are correlated on all sides of this equation. Once again, a unitary representation implies that the right-hand side of this expression should be greater than zero, leaving us with the celebrated BPS bound:

$$M \geq |Z_{\bar{I}}| \tag{11}$$

for all $\bar{I}$. Note that if the bound is saturated and $M = |Z_{\bar{I}}|$, then one of the supercharges $\mathcal{Q}^{\bar{I}}_+$ or $\mathcal{Q}^{\bar{I}}_-$ kills the state, implying that this particular representation is smaller; we say that BPS states live in *short* multiplets.

Why is this useful? Suppose we find a short multiplet with $M = |Z_{\bar{I}}|$. Then for M to stray from this special value (say under a variation of a parameter of the theory, a coupling constant etc.), the short multiplet must become long, which means that it must pair up with its conjugate short multiplet. Typically this is not possible and the mass of the state is restricted to be $|Z_{\bar{I}}|$ always.[1]

With no further ado we present a table of the various possible states for massive supermultiplets:

[1] Note further that because Z is a central element of the superalgebra, which includes $\hat{P}_\mu$, it is actually a c-number constant, rather than an operator like $\hat{P}_\mu$. This makes the fact that Z is determined by the algebra much more powerful. Thanks to Michael Kiermaier for emphasizing this distinction.

Helicity	$\mathcal{N}=1$ gauge	$\mathcal{N}=1$ chiral	$\mathcal{N}=2$ gauge	$\mathcal{N}=2$ BPS gauge	$\mathcal{N}=2$ BPS hyper	$\mathcal{N}=4$ BPS gauge
1	1	0	1	1	0	1
$\frac{1}{2}$	2	1	4	2	(1+1)	4
0	1	2	6	1	(2+2)	6

Here all states will be present with their CPT conjugates of opposite helicity. Note that the massive gauge multiplet contains precisely the same states as (massless chiral + massless gauge) multiplets. Compare this to the familiar non-supersymmetric construction via the Higgs mechanism of a massive vector boson from (massless scalar + massless gauge) bosons. It should not be surprising that the supersymmetric version of this is called the Super-Higgs mechanism.

2 $\mathcal{N}=4$ Spectrum

We now turn to a closer examination of the spectrum for the maximally supersymmetric $\mathcal{N}=4$ case, which can be formulated with any gauge group (but we will restrict ourselves to $U(N)$). Here the field content is (compare with the massless $\mathcal{N}=4$ gauge multiplet in the table):

1. Gauge field A_μ: $\mu = 0...3$ is a Lorentz vector index.
2. Weyl spinors λ^I_α: $I = 1...(\mathcal{N}=4)$, α a spinor index.
3. Scalars X^i: here $i = 1...6$, as will be explained shortly.

As the gauge field is necessarily in the adjoint of the gauge group, all of these fields are $N \times N$ matrices in the adjoint of $U(N)$. Note that the R-symmetry is $SU(4)$ (the diagonal $U(1)$ is broken, for reasons which we will presumably learn later), which acts on the fermions λ^I in the fundamental of $SU(4)$. To understand the action on the scalars, it is useful to know that

$$SU(4) \sim SO(6) \tag{12}$$

The fundamental of $SU(4)$ is the spinor representation of $SO(6)$. As we obtained the X's by the action of two antisymmetrized Q's on the ($SU(4)$ singlet) A_μ, we expect it to transform in the antisymmetric tensor representation of $SU(4)$. This works out to be the standard vector 6 of $SO(6)$, which acts on the i index on X^i.

If we stray into string theory briefly, we see that this theory is also the worldvolume theory on a stack of N $D3$ branes. This gives us another way to understand the spectrum. Imagine placing a single $D3$ brane in $\mathbb{R}^{3,1}$. This breaks 10 (total) $-$ 4 (along the brane) = 6 translational symmetries, and so we expect to have 6 Goldstone bosons in the worldvolume theory of the $D3$ branes–these are the X^i's! Similarly, the full ten-dimensional theory had 32 supercharges (from $\mathcal{N}=2$ supergravity in ten dimensions), but we only see 16 supercharges remaining in the $\mathcal{N}=4$ theory. To understand where they went, consider the supercharges Q:

$$\{Q, Q\} \sim \gamma^\mu P_\mu \tag{13}$$

Those Q whose anticommutators generate translations along the 6 broken translational symmetries are broken, and this leaves us with 16 unbroken supersymmetries. By "Goldstino's theorem", this breaking results in 16 massless fermions or "goldstinos" ('goldstini'?), which can be rearranged into the $4\lambda^I$'s.

Finally, we note that another way to produce this theory is to start with maximal superymmetry in $d = 10$ *without* gravity. This is a pure $\mathcal{N} = 1$ gauge multiplet, and contains 16 supercharges; we are no longer allowed to break these supersymmetries with D3 branes (since we need all of them!). What we *can* do is consider field configurations that are independent of the 6 extra dimensions and simply integrate over them in the 10 dimensional action. This procedure is called "dimensional reduction" and will give us a 4-dimensional action containing precisely the field content of $\mathcal{N} = 4$ SYM.