In today’s lecture we will talk about:

1. AdS wave equation near the boundary.

2. Masses and operator dimensions: $\Delta(\Delta - D) = m^2 L^2$.

Erratum: The massive geodesic equation $\ddot{x} + \Gamma \dot{x} \dot{x} = 0$ assumes that the dot differentiates with respect to proper time.

Recap: Consider a scalar in AdS$p+2$ (where $p + 1$ is the number of spacetime dimensions that the field theory lives in). Let the metric be:

$$ds^2 = L^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2},$$ (1)

then the action takes the form:

$$S[\phi] = -\frac{\kappa}{2} \int d^{p+1}x \sqrt{g} (\left(\partial \phi\right)^2 + m^2 \phi^2 + b\phi^3 + ...),$$ (2)

where $(\partial \phi)^2 \equiv g^{AB} \partial_A \phi \partial_B \phi$ and $x^A = (z, x^\mu)$. Our goal is to evaluate:

$$\ln\langle \exp -\int d^D x \phi_0 O \rangle_{CFT} = \text{extremum}_{[\phi_0]} S[\phi_0]$$, (3)

where $S[\phi] \equiv S[\phi^*(\phi_0)] \equiv W[\phi_0]$, i.e. by using the solution to the equation of motion subject to boundary conditions. Now Taylor expand:

$$W[\phi_0] = W[0] + \int d^D x \phi_0(x) G_1(x) + \frac{1}{2} \int \int d^D x_1 d^D x_2 \phi_0(x_1) \phi_0(x_2) G_2(x_1, x_2) + ...$$ (4)

where

$$G_1(x) = \langle O(x) \rangle = \frac{\delta W}{\delta \phi_0(x)}|_{\phi_0=0},$$ (5)

$$G_2(x) = \langle O(x_1) O(x_2) \rangle_c = \frac{\delta^2 W}{\delta \phi_0(x_1) \delta \phi_0(x_2)}|_{\phi_0=0}.$$ (6)
Now if there is no instability, then $\phi_0$ is small and so is $\phi$, so you can ignore third order terms in $\phi$. From last time:

$$S[\phi] = \frac{\kappa}{2} \int_{\text{AdS}_{p+2}} d^{p+2}x \sqrt{g} [\phi (-\nabla^2 + m^2) \phi + \mathcal{O}(\phi^3)] - \frac{\kappa}{2} \int_{\partial \text{AdS}} d^{p+1}x \sqrt{\gamma} \phi (n, \partial) \phi,$$

where the last term is the boundary action, $n$ is a normalized vector perpendicular to the boundary and

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B).$$

Now if the scalar field satisfies the wave equation:

$$(-\nabla^2 + m^2)\phi^* = 0,$$

$$W[\phi_0] = S_{\text{bdy}}[\phi^*[\phi_0]],$$

then we can use translational invariance in $p + 1$ dimensions, $x^\mu \to x^\mu + a^\mu$, in order to Fourier decompose the scalar field:

$$\phi(z, x^\mu) = e^{ik.x} f_k(z).$$

Now, substituting (11) into (9) and assuming that the metric only depends on $z$ we get:

$$0 = (g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{g}} \partial_z (\sqrt{g} g^{zz} \partial_z) + m^2) f_k(z)$$

$$= \frac{1}{L^2} [z^2 k^2 - z^{D+1} \partial_z (z^{-D+1} \partial_z) + m^2 L^2] f_k,\quad \text{(13)}$$

where we have used $g^{\mu\nu} = (z/L)^2 \delta^{\mu\nu}$. The solutions of (12) are Bessel functions but we can learn a lot without using their full form. For example, look at the solutions near the boundary (i.e. $z \to 0$). In this limit we have power law solutions, which are spoiled by the $z^2 k^2$ term. Try using $f_k = z^\Delta$ in (12):

$$0 = k^2 z^{2+\Delta} - z^{D+1} \partial_z (\Delta z^{-D+\Delta}) + m^2 L^2 z^\Delta$$

$$= (k^2 z^2 - \Delta (\Delta - D) + m^2 L^2) z^\Delta,\quad \text{(15)}$$

and for $z \to 0$ we get:

$$\Delta (\Delta - D) = m^2 L^2$$

The two roots for (16) are

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2 L^2}.$$
Comments

- The solution proportional to $z^{\Delta_-}$ is bigger near $z \to 0$.
- $\Delta_+ > 0 \forall m$, therefore $z^{\Delta_+}$ decays near the boundary.
- $\Delta_+ + \Delta_- = D$.

Next, we want to improve the boundary conditions that allow solutions, so take:

$$\phi(x, z)|_{z=\epsilon} = \phi_0(x, \epsilon) = \epsilon^{\Delta_-} \phi_{0}^{\text{Ren}}(x), \quad (18)$$

where $\phi_0^{\text{Ren}}$ is the renormalized field. Now with this boundary condition, $\phi(z, x)$ is finite when $\epsilon \to 0$, since $\phi_0^{\text{Ren}}$ is finite in this limit.

Wavefunction renormalization of $O$ (Heuristic but useful)

Suppose:

$$S_{\text{bdy}} \ni \int_{z=\epsilon} d^{D+1} x \sqrt{\gamma} \phi_0(x, \epsilon) O(x, \epsilon) \quad (19)$$

$$= \int d^D x \left( \frac{L}{\epsilon} \right)^D (\epsilon^{\Delta_-} \phi_0^{\text{Ren}}(x)) O(x, \epsilon), \quad (20)$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^D$. Demanding this to be finite as $\epsilon \to 0$ we get:

$$O(x, \epsilon) \sim \epsilon^{D-\Delta_-} O^{\text{Ren}}(x) \quad (21)$$

$$= \epsilon^{\Delta_+} O^{\text{Ren}}(x), \quad (22)$$

where in the last line we have used $\Delta_+ + \Delta_- = D$. Therefore, the scaling of $O^{\text{Ren}}$ is $\Delta_+ \equiv \Delta$.

Comments

- We will soon see that $\langle O(x) O(0) \rangle \sim \frac{1}{|x|^{2\Delta}}$.
- We had a second order ODE, therefore we need two conditions in order to determine a solution (for each $k$). So far we have imposed:
  1. For $z \to \epsilon$, $\phi \sim z^{\Delta_-} \phi_0 + \text{terms subleading in } z$. Now we will also impose
  2. $\phi$ regular in the interior of AdS (i.e. at $z \to \infty$).

Comments on $\Delta$

1. The $\epsilon^{\Delta_-}$ factor is independent of $k$ and $x$, which is a consequence of a local QFT (this fails in exotic examples).
2. **Relevantness:** Since \( m^2 > 0 \implies \Delta \equiv \Delta_+ > D \), so \( O_\Delta \) is an irrelevant operator. This means that if you perturb the CFT by adding \( O_\Delta \) to the Lagrangian, then:

\[
\Delta S = \int d^Dx \text{(mass)}^{D-\Delta} O_\Delta,
\]

where the exponent is negative, so the effects of such an operator go away in the IR. For example, consider a dilaton mode with \( l > 0 \), its mass is given by (for \( D = 4 \)):

\[
m^2 = \frac{(l+4)!}{L^2}.
\]

The operator corresponding to this is:

\[
\text{tr}(F^2 X^{i_1 \ldots i_l}),
\]

with \( \Delta = 4 + l > D \), therefore it is an irrelevant operator. Now consider a dilaton mode with \( l = 0 \): then \( m^2 = 0 \), therefore, \( \Delta = D \) and hence it corresponds to a marginal operator (an example of such operator is the Lagrangian). If \( m^2 < 0 \), then \( \Delta < D \), so it corresponds to a relevant operator, but it is ok if \( m^2 \) is not too negative ("Breitenlohner - Freedmasn (BF) - allowed tachyons" with \(-|m_{BF}|^2 \equiv -(D/2L)^2 < m^2\)).

3. **Instability:** This occurs when a renormalizable mode grows with time without a source. But in order to have \( S[\phi] < \infty \), the solution must fall off at the boundary. This requires a gradient energy that \( \sim \frac{1}{z^2} \). Note:

\[
\Delta_{\pm} = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2L^2}.
\]

If:

\[
m^2L^2 < \left(\frac{D}{2}\right)^2 \equiv -|m_{BF}|^2,
\]

then \( \Delta_{\pm} \) is complex, therefore we have \( \Delta = D/2 \), which is larger than the unitary bound. In this case, \( \phi \sim z^\Delta \) decays near the boundary (i.e. in the UV). In order to see the instability that occurs when \( m^2L^2 < \left(\frac{D}{2}\right)^2 \) more explicitly, rewrite (9) as a Schrodinger equation, by writing \( \phi(z) = A(z)\psi(z) \), where we choose \( A(z) \) in order to remove the first derivative of \( \psi(z) \). Then, equation (9) becomes:

\[
(-\partial_z^2 + V(z))\psi(z) = E\psi(z),
\]

where \( E = \omega^2 - k^2 \), \( V(z) = \sigma/z^2 \) and \( \sigma = m^2L^2 - (D^2 - 1)/4 \). An instability occurs when \( E < 0 \), i.e. \( \omega^2 < 0 \) and hence \( \phi \sim e^{i\omega t}\phi(z) = e^{+|\omega|t}\phi(z) \) grows with time. Now the claim is that \( V = \sigma/z^2 \) has no negative energy states if \( \sigma > -1/4 \). Note that the notion of normalizability here and before are related (Pset 4):

\[
||\psi||^2 = \int dz \psi^\dagger \psi < \infty,
\]

and \( S[\phi] = \int dz \sqrt{g} ((\partial \phi)^2 + m^2) \)

4. The formula we found before (expression (16)) depends on the spin. For a \( j \)-form in AdS we have:

\[
(\Delta + j)(\Delta + j - D) = m^2L^2.
\]
For example, for $A_\mu$ massless we have:

$$\Delta(j^\mu) = D - 1 \rightarrow \text{conserved},$$  \hspace{1cm} (32)

for $g_{\mu\nu}$ massless we have:

$$\Delta(T^{\mu\nu}) = D \rightarrow \text{required from CFT.}$$  \hspace{1cm} (33)