8.821 F2008 Lecture 14: Wave equation in AdS, Green's function

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Topics for this lecture

- Find $\phi^{[\phi_0]}(x)$ by Green functions in x-space (efficient)
- Compute $\langle \mathcal{O}\mathcal{O} \rangle$, counter terms
- Redo in p-space (general)

References

- Witten, hep-th/9802150
- GKP, hep-th/9802109

Solving Wave Equation I (Witten's method)

Let's study the wave equation in AdS in some detail. This first method uses a trick by Witten which is efficient but slightly obscure.

If we know "bulk-to-boundary" Green's function K regular in the bulk, such that

$$(-\Box + m^2) K_p(z,x) = 0 \tag{1}$$
$$K_p(z,x) \to \epsilon^{\Delta_-} \delta^D_\epsilon(x-p), \quad z \to \epsilon \tag{2}$$

where p is some point on the boundary,
then the field in the bulk

$$\phi^{[\phi_0]}(z,x) = \int d^D x' \, \phi_0^{\text{Ren}}(x') K_{x'}(z,x) \to z^{\Delta_-} \phi_0^{\text{Ren}}(x)$$

solves (1).

Euclidean AdS

Recall the metric on AdS with curvature scale L in the upper half plane coordinates:

$$ds^2 = L^2 \frac{dz^2 + dx^2}{z^2}$$

Now here comes some fancy tricks, thanks to Ed:

Trick (1): Pick $p =$ "point at ∞". This implies that the Green's function $K_\infty(z, x)$ is x-independent.

The wave equation at $k = 0$:

$$0 = \left[-z^{D+1}\partial_z z^{-D+1}\partial_t + m^2\right] K_\infty(z)$$

can easily be solved. The solution is power law (recall that in the general-k wave equation, it was the terms proportional to k^2 that ruined the power-law behavior away from the boundary)

$$K_\infty(z) = c_+ z^{\Delta_+} + c_- z^{\Delta_-}$$

We can eliminate one of the constants: $c_- = 0$, whose justification will come with the result.

Trick (2): Use AdS isometries to map $p = \infty$ to finite x. Let $x^A = (x^\mu, z)$, take $x^A \to (x')^A = x^A/(x^B x_B)$. The inversion of this mapping is:

$$I : \begin{cases} x^\mu \to \frac{x^\mu}{z^2+x^2} \\ z \to \frac{z}{z^2+x^2} \end{cases}$$

Claim: I

A) is an isometry of AdS (also Minkowski version, see pset 4)

B) is not connected to **1** in SO(D,2)

C) maps $p = \infty$ to $x = 0$, i.e., $I : K_\infty(z, x) \to K_\infty(z', x') = K_0(z, x) = c_+ z^{\Delta_+}/(z^2 + x^2)^{\Delta_+}$.

Some notes:

(i) That this solves the wave equation (1) as neccessary can be checked directly.

(ii) The Green's function is

$$K_{x'}(z, x) = c_+ \frac{z^{\Delta_+}}{(z^2 + (x - x')^2)^{\Delta_+}} \equiv K(z, x; x')$$

(iii) The limit of the Green's function as $z \to 0$, *i.e.* the boundary is

$$K(z, x; x') \to \begin{cases} c z^{\Delta_+} \to 0, & \text{if } x \neq x' \\ c z^{-\Delta_+} \to \infty, & \text{if } x = x' \end{cases}$$

(recall that $\Delta_+ > 0$ for any D, m). More specifically, the Green's function approaches a delta function:

$$K(z,x;x') \to \text{const} \cdot \epsilon^{\Delta_-} \delta^D(x-x').$$

Clearly it has support only near $x = x'$, but to check this claim we need to show that it has finite measure:

$$\begin{aligned} \int d^D x\, \epsilon^{-\Delta_-} K_0(\epsilon, x) &= \int d^D x \frac{c\epsilon^{\Delta_+ - \Delta_-}}{(\epsilon^2 + x^2)^{\Delta_+}} \\ &= \frac{c\epsilon^D \epsilon^{2\Delta_+ - D}}{\epsilon^{2\Delta_+}} \int d^D \bar{x} \frac{1}{(1+\bar{x}^2)^{\Delta_+}} \\ &= c\frac{\pi^{\frac{D}{2}} \Gamma(\Delta_+ - \frac{D}{2})}{\Gamma(\Delta_+)}. \end{aligned}$$

We will choose the constant c to set this last expression equal to one. Hence,

$$\begin{aligned} \phi^{[\phi_0]}(z,x) &= \int d^D x'\, K_{x'}(z,x) \phi_0^{\text{Ren}}(x') \\ &= \int d^D x' c \frac{x^{\Delta_+}}{(z^2 + (x-x')^2)^{\Delta_+}} \phi_0^{\text{Ren}}(x') \ ; \end{aligned}$$

this solves (1) and approaches $\epsilon^{\Delta_-} \phi_0^{\text{Ren}}(x)$ as $z \to \epsilon$.

The action is related to expectation values of operators on the boundary:

$$\begin{aligned} S\left[\phi^{[\phi_0]}\right] &= -\ln \langle e^{-\int \phi_0 \mathcal{O}} \rangle \\ &= -\frac{\eta}{2} \int_{\partial AdS} \sqrt{\gamma}\, \phi\, n \cdot \partial \phi \\ &= -\frac{\eta}{2} \int d^D x \sqrt{g}\, g^{zz} \phi(z,x) \partial_z \phi(z,x) \Big|_{z=\epsilon} \\ &= -\frac{\eta}{2} \int d^D x_1 d^D x_2\, \phi_0^{\text{Ren}}(x_1) \phi_0^{\text{Ren}}(x_2) \mathcal{F}_\epsilon(x_1, x_2) \end{aligned}$$

where the "flux factor" is

$$\mathcal{F}_\epsilon(x_1,x_2) \equiv \int d^D x \frac{K(z,x;x_1) z\partial_z K(z,x;x_2)}{z^D} \Big|_{z=\epsilon}.$$

The boundary behavior of K is:

$$K^{\Delta_+}(z,x;x')\Big|_{z=\epsilon} = \epsilon^{\Delta_-} \left(\delta_\epsilon^D(x-x') + \mathcal{O}(\epsilon^2) \right) + \epsilon^{\Delta_+} \left(\frac{c}{(x-x')^{2\Delta_-}} + \mathcal{O}(\epsilon^2) \right)$$

the first terms sets: $c^{-1} = \pi^{\frac{D}{2}} \Gamma(\Delta_+ - \frac{D}{2}) / \Gamma(\Delta_+)$, the second term is subleading in z.

$$z\partial_z K(z,x;x')\Big|_{z=\epsilon} = \Delta_+ \epsilon^{\Delta_-} \delta(x-x') + \Delta_+ c z^{\Delta_+} \frac{1}{(x-x')^{2\Delta_+}} + \ldots$$

Ok, now for the 2-point correlation function on the boundary:

$$G_2(x_1,x_2) \equiv \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_c = \frac{\delta}{\delta\phi_0(x_1)} \frac{\delta}{\delta\phi_0(x_2)} \left(-S\left[\phi^{[\phi_0]}\right] \right) = \eta \mathcal{F}_\epsilon(x_1,x_2).$$

We must be careful when evaluating the cases $x_1 \neq x_2$ and $x_1 = x_2$, which we do in turn.

Firstly, if $x_1 \neq x_2$:

$$\begin{aligned} G_2(x_1 \neq x_2) &= \frac{\eta}{2}\int_{z=\epsilon} d^D x z^{-D}\left(z^{\Delta_-}\delta^D(x-x_1)+\mathcal{O}(z^2)\right)\left((\text{ignore by}\, x_1\neq x_2)+\frac{\Delta_+ c z^{\Delta_+}}{(x_1-x_2)^{2\Delta_+}}+\mathcal{O}(z^2)\right) \\ &= \frac{\eta}{2}c\Delta_+\epsilon^{-D+\Delta_-+\Delta_+}\frac{1}{(x_1-x_2)^{2\Delta_+}}+\mathcal{O}(\epsilon^2) \\ &= \frac{\eta c\Delta_+}{2(x_1-x_2)^{2\Delta_+}}. \end{aligned}$$

Good. This is the correct form for a two point function of a conformal primary of dimension Δ_+ in a CFT; this is a check on the prescription.

Secondly, if $x_1 = x_2$:

$$G_2(x_1,x_2)=\eta\left(\Delta_-\epsilon^{2\Delta_- - D}\delta^D(x_1-x_2)+\frac{c\Delta_+}{(x_1-x_2)^{2\Delta_+}}+\Delta_+c^2\epsilon^{2\Delta_+-D}\int d^Dx\frac{1}{(x-x_1)^{2\Delta_+}(x-x_2)^{2\Delta_+}}\right)$$

As $\epsilon\to 0$, the first term is divergent, the second term is finite, and the third term vanishes. The first term is called a "divergent contact term". It is scheme-dependent and useless.

Remedy: Holographic Renormalization. Add to S_{geometry} the contact term

$$\begin{aligned} \Delta S = S_{\text{c.t.}} &= \frac{\eta}{2}\int_{\text{bdy}} d^Dx\left(-\Delta_-\epsilon^{2\Delta_- - D}\left(\phi_0^{\text{Ren}}(x)\right)^2\right) \\ &= -\Delta_-\frac{\eta}{2}\int_{\partial AdS, z=\epsilon}\sqrt{\gamma}\,\phi^2(z,x). \end{aligned}$$

Note that this doesn't affect the equations of motion. Nor does it affect $G_2(x_1\neq x_2)$.

Solving Wave Equation II (k-space)

Since the previous approach isn't always available (for example if there is a black hole in the spacetime), let's redo the calculation in k-space.

Return to wave equation

$$0=\left[z^{D+1}\partial_z\left(z^{-D+1}\partial_z\right)-m^2L^2-z^2k^2\right]f_k(z)$$

with $k^2=-\omega^2+\mathbf{k}^2>0$. The solution is

$$f_k(z)=A_K z^{\frac{D}{2}}K_\nu(kz)+A_I z^{\frac{D}{2}}I_\nu(kz),$$

with $\nu=\sqrt{(D/2)^2+m^2L^2}=\Delta_+-D/2$. Assume $k\in\mathbb{R}$ (real time issues later). As $z\to\infty$: $K_\nu\sim e^{-kz}$ and $I_\nu\sim e^{kz}$. The latter is not okay, so $A_I=0$.

At boundary:

$$K_\nu(n) \sim n^{-\nu}\left(a_0 + a_1 n^2 + a_2 n^4 + \ldots\right) + \begin{cases} n^\nu (b_0 + b_1 n^2 + b_2 n^4 + \ldots),\ \nu \notin \mathbb{R} \\ n^\nu \ln n\, (b_0 + b_1 n^2 + b_2 n^4 + \ldots),\ \nu \in \mathbb{R} \end{cases}$$

Hence

$$f_k(z) = A_K z^{D/2} K_\nu(kz) \sim z^{\frac{D}{2}\pm\nu} = z^{\Delta_\pm}, \quad \text{as } z \to 0.$$