

8.821 F2008 Lecture 18: Wilson Loops

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1 Minimum Surfaces

The expectation value of Wilson loop operators $\langle W[\mathcal{C}] \rangle$ in the CFT can be computed by computing the string theory partition function on AdS with a string world sheet ending on the loop \mathcal{C} . That is,

$$\langle W[\mathcal{C}] \rangle_{CFT} = Z_{\text{string theory}}[\text{string ending on } \mathcal{C}]$$

Further we saw that, at large λ and large N , the world sheet path integral is dominated by the saddle point. In the saddle point approximation, the string theory partition function is given by,

$$Z_{\text{string theory}} \sim \exp \left[- \text{extremum}_{\partial\Sigma=\mathcal{C}} \left(\frac{\text{Area}(\Sigma)}{\alpha'} \right) \right].$$

Hence, the computation of $\langle W[\mathcal{C}] \rangle$ in the CFT can be mapped to finding a minimum surface in AdS . The area of string worldsheet (classical) is given by,

$$\text{Area}[X] = \int d\sigma d\tau \sqrt{\det \partial_\alpha X^A \partial_\beta X^B g_{AB}(X)} = \int d\sigma d\tau \sqrt{\frac{L^4}{z^4} \det \partial_\alpha X^A \partial_\beta X^B \delta_{AB}}$$

Here we have used the fact that $g_{AB} = \frac{L^2}{z^2} \delta_{AB}$ for Euclidean AdS . Notice that,

$$\ln \langle W \rangle = \frac{L^2}{\alpha'} \left(\text{extremum}_{\partial\Sigma=\mathcal{C}} \text{Area}(\Sigma) \right)_{L=1} = \sqrt{\lambda} \left(\text{extremum}_{\partial\Sigma=\mathcal{C}} \text{Area}(\Sigma) \right)_{L=1}$$

Further, the area of worldsheet is infinite. This divergence arises from the fact that the worldsheet goes all the way to the boundary of AdS . Let us zoom into a region close to the boundary and introduce a cut-off at $z = \epsilon$ which is equivalent to the UV cut-off of the field theory. We can think of the direction along the curve \mathcal{C} as the τ coordinate of the worldsheet near $z = \epsilon$ and σ extends along the radial direction near $z = \epsilon$ (please see Fig. 1). The area evaluated upto the cut-off ϵ has the following behavior,

$$A_\epsilon = \int_\epsilon dz \int_{\mathcal{C}} d\tau \frac{1}{z^2} \sqrt{\det \partial_\alpha X^A \partial_\beta X^B \delta_{AB}} \sim \frac{\text{length}[\mathcal{C}]}{\epsilon} + \dots$$

It can be seen that A diverges as ϵ^{-1} . Let us compare this result with Wilson loop calculation in the field theory. In the field theory, the Wilson loop is the trajectory of a heavy quark. Quarks couple

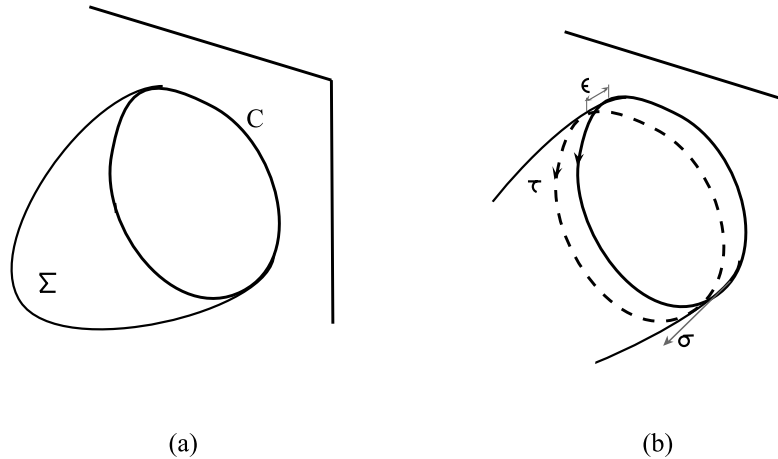


Figure 1: (a) Worldsheet ending on curve \mathcal{C} (b) Worldsheet near the boundary.

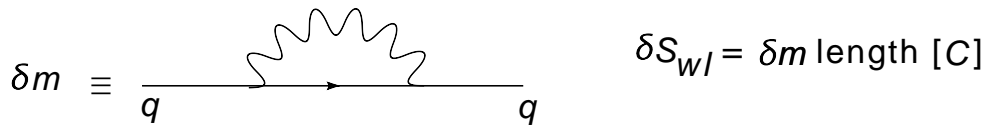


Figure 2: Diagram contributing to Mass Renormalization of heavy quark

to gluons and they can emit and reabsorb gluons. An example of such a process is the leading one-loop correction to the quark propagator (shown in Fig. 2). This diagram corresponds to the mass renormalization (δm_q) of the heavy quark at zero quark momentum. This implies we must add $\delta m_q \int ds$ to the worldline action of the quark. So, the ϵ^{-1} divergence in the Wilson loop can be absorbed into mass of the quark. Hence, the prescription to calculate the renormalized wilson loop is to subtract off the ϵ^{-1} divergence in the extremum area of the worldsheet. This prescription has a geometrical interpretation too! The divergence in the world sheet area is the area of a tube (shape of the cross-section perpendicular to the radial direction is independent of z) in AdS, whose base is the \mathcal{C} . Subtracting the ϵ^{-1} divergence is equivalent to subtracting the area of this tube from the area of the minimum surface.

Note that this is not true for the BPS case as, there is no mass renormalization of a BPS particle. There are two diagrams that contribute to the mass renormalization (shown in Fig .3)- a diagram involving gauge boson and another diagram involving the scalars. BPS case corresponds to some correlated choice of the trajectory in spacetime and the trajectory in the S_5 . If there is some constraint on these trajectories, the loop can be BPS and the two contributions cancel. On the other hand, the area of the worldsheet still has some divergences. The resolution of this puzzle



Figure 3: Diagrams contributing to Mass renormalization of a BPS particle

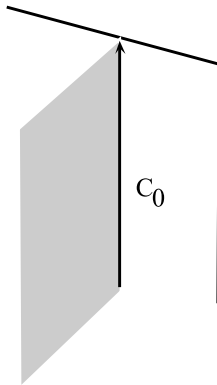


Figure 4: Minimum surface ending on a straight line

is that the expectation value of the Wilson loop is not the area but its Legendre transformation.¹ We shall now look at some examples of curves for which we can say something about the Wilson loops.

2 Examples

2.1 \mathcal{C}_0 : Straight Line

In this case the minimal surface is a flat surface that ends on the straight line (Please see Fig. 4). It is obvious that a flat surface satisfies the equation of motion. The area of this fat surface is $\frac{1}{\epsilon} \int d\tau$. Further, note that the flat surface is a surface whose "cross-section" doesn't depend on the radial coordinate. Hence, the renormalized Wilson loop is obtained by subtracting the area of the flat surface from itself. That is,,

$$\ln\langle W[\mathcal{C}_0] \rangle_{\text{ren}} = \text{Area of flat surface} - \text{Area of flat surface} = 0 \implies \langle W[\mathcal{C}_0] \rangle = 1$$

In the next example, we shall see how we can use this trivial result to compute the Wilson loop for a circular contour.

2.2 \mathcal{C}_1 : Circle

There are two ways by which we can find the minimal surface - (a) solve the equation of motion and (b) make use of conformal symmetry. In this example, we will calculate the Wilson loop of a circular contour by mapping the circle to a straight line using conformal transformations. The transformation that takes a line to a circle at the boundary is given by,

$$x^i = \frac{x^i + b^i x^2}{1 + 2b_i x^i + b^2 x^2}$$

¹Please see N. Drukker, D. J. Gross and H. Ooguri (hep-th/9904191) for a detailed discussion.

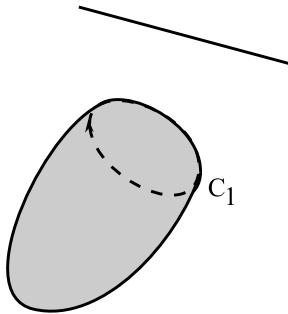


Figure 5: Minimum surface ending on a circle

Let us assume $b^3 = 0, b^4 = 0$ for simplicity. Then, the above transformation maps the line $\mathcal{C}_0 = \{\{x_i\} | x_2 = 0, x_3 = 0, x_4 = 0\}$ to the circle $\mathcal{C}_1 = \{\{x_i\} | x_2 + b_2^2(x_2^2 + x_1^2) = 0, x_3 = 0, x_4 = 0\}$. The radius of the circle \mathcal{C}_1 is $(2b_2)^{-2}$ and the centre of this circle is located at $(0, (2b_2)^{-1}, 0, 0)$. Let us denote the radius of the circle by a . The special conformal transformation symmetry of the boundary theory extends to an isometry of the AdS which is given by,

$$x^i = \frac{x^i + b^i(x^2 + z^2)}{1 + 2b_i x^i + b^2 x^2}$$

$$z = \frac{z}{1 + 2b_i x^i + b^2 x^2}$$

Using the above transformation on the trivial solution in the previous example, we find that the solution for the minimum surface (in the parametric form) is given by,

$$\vec{x} = \sqrt{a^2 - z^2}(\cos \theta, \sin \theta, 0, 0), \quad -a \leq z \leq a$$

One can check that this solves the equation of motion. The worldsheet can be parametrized by $z(\sigma)$ and $\theta(\tau)$. The induced metric has the following form,

$$ds^2 = \frac{L^2}{z^2} \left((a^2 - z^2) d\theta^2 + \frac{a^2}{a^2 - z^2} dz^2 \right)$$

Plugging the above solution into the expression for worldsheet area (Nambu-Goto action) get,

$$S_{NG}[X] = \frac{\text{Area}}{2\pi\alpha'} = \frac{L^2}{2\pi\alpha'} \int_0^{2\pi} d\theta \int_\epsilon^a dz \frac{a}{z^2} = \frac{L^2}{\alpha'} \left[\frac{a}{\epsilon} - 1 \right]$$

The ϵ^{-1} divergence is cancelled by the area of the cylinder with circular base (shown in Fig. 6). Hence,

$$\ln \langle W[\mathcal{C}_1] \rangle_{\text{ren}} = -S_{NG} + \text{Area of cylinder} = \frac{L^2}{\alpha'} \implies \langle W[\mathcal{C}_1] \rangle = e^{\sqrt{4\pi\lambda}}$$

Note that, the expectation value of the Wilson loop is independent of a . This is a consequence of scale invariance. Further, we can see that the expectation value of this Wilson loop is not 1, even though it is related to the previous example by a symmetry. This is because, the whole contribution to the Wilson loop comes from the anomaly in conformal transformation at the point at infinity. The expectation value of the Wilson loop obtained here is the large λ limit of the exact answer $\sim I_1(\sqrt{\lambda})$.²

²Please see hep-th/0010274.

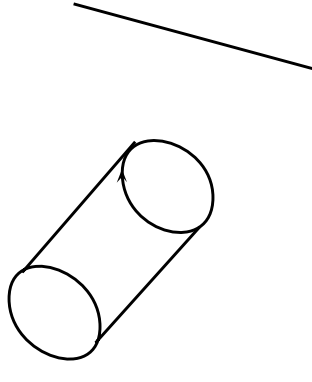


Figure 6: "Tube" ending on a circle

2.3 Quark-Antiquark Potential

In this example we shall calculate the potential between a static quark and a static anti-quark. Figure (7) shows the configuration used to calculate the quark-antiquark potential. The vertical direction in Figure (7 a) denoted by t is taken as the real time direction and nothing depends on t . The quark and the anti-quark are separated by a distance r in the x direction. We may further assume that they are located at the same point in S_5 . This time-independent configuration at the boundary suggests that we can look for time-independent solution for the minimum surface. The area will have divergence arising from the integral over the time-direction. This can be regulated by introducing a cut-off (T). The goal is to find the area per unit time of the minimum surface ending on this configuration. To calculate this, it is useful to work in the static gauge, in which,

$$\tau = t \quad \sigma = x \quad \text{and} \quad z = z(x)$$

The induced metric in AdS in this parametrization of the world sheet is given by,

$$ds^2 = \frac{L^2}{z^2} \left[dt^2 + dx^2 \left(1 + \left(\frac{\partial z}{\partial x} \right)^2 \right) \right] = \gamma_{ab} dx^a dx^b$$

The Nambu-Goto action is then given by,

$$S = \frac{1}{2\pi\alpha'} \int_{-T/2}^{T/2} dt \int_{-r/2}^{r/2} dx (\det[\gamma])^{1/2}$$

In the above expression, the integral over time is cut-off by T . The above expression simplifies to,

$$S = \sqrt{4\pi\lambda T} \int_{-r/2}^{r/2} dx \frac{\left(1 + (\partial_x z)^2 \right)^{1/2}}{z^2(x)}$$

Finding, the minimum surface in this case is a classical mechanics problem in one variable. This variational problem can be solved by making use of the translation symmetry ($\rightarrow x + a$) of the Lagrangian. There is a conserved charge (independent of x) associated with this symmetry which is given by,

$$\mathcal{H} = \mathcal{L} - z' \frac{\partial \mathcal{L}}{\partial z'}$$

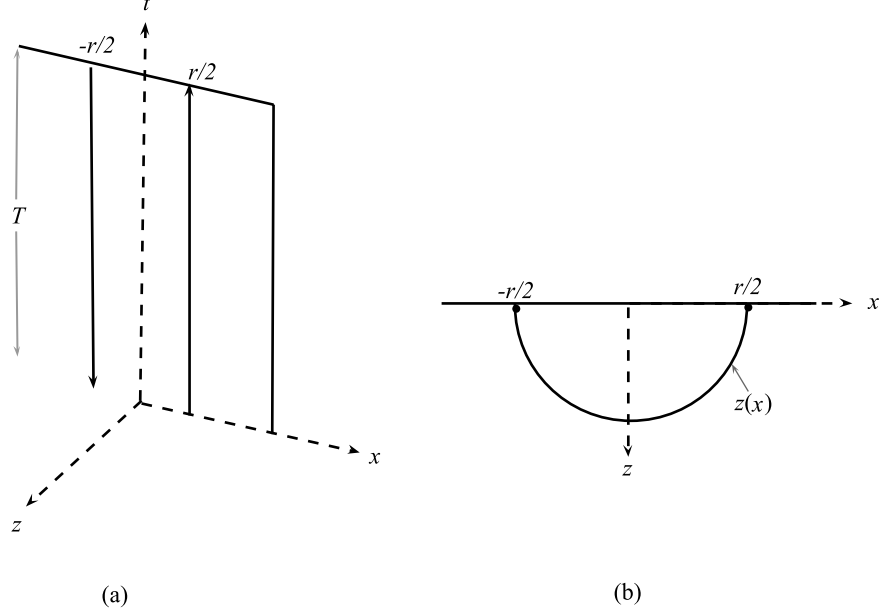


Figure 7: (a) Time Independent Boundary Configuration (b) Top-view of the configuration

In the above expression, \mathcal{L} is the Lagrangian, \mathcal{H} is the conserved current and z' denotes the derivative of z with respect to x . Hence,

$$\frac{1}{z^2\sqrt{1+z'^2}} = \mathcal{H} = \text{constant}$$

Note that, z is an even function of x . This implies z is maximum at $x = 0$. Using this condition and the above equation, we can now solve for x as a function of z . The solution is given by,

$$x = \pm \int_{z_{max}}^z \mathcal{H} d\tilde{z} \frac{\tilde{z}^2}{\sqrt{1 - \mathcal{H}^2 \tilde{z}^4}}$$

Note that, the integrand becomes imaginary when $\tilde{z} > \sqrt{1/\mathcal{H}}$. Hence, the maximum value that z can take is $\sqrt{1/\mathcal{H}}$. Let us substitute $\tilde{z} = y\mathcal{H}^{-1/2}$ in the expression for the integrand. The solution for $x > 0$ can be written as,

$$x = z_{max} \int_{z/z_{max}}^1 dy \frac{y^2}{\sqrt{1-y^4}}$$

We can now solve for z_{max} in terms of r using the fact that $x = r/2$ when $z = 0$. Substituting this condition in the above expression we get,

$$r/2 = z_{max} \int_0^1 dy \frac{y^2}{\sqrt{1-y^4}} = z_{max} \frac{\sqrt{2\pi^3}}{[\Gamma(1/4)]^2}$$

Hence, we can express our solution as $x = X(r, z)$. Substituting this solution in Nambu-Goto action we get,

$$S_{NG} = T\sqrt{4\pi\lambda} \int_{-r/2}^{r/2} \frac{\sqrt{1+z'^2}}{z^2} dx$$

Using the fact that $dz/dx = \sqrt{((z_{max}/z)^4 - 1)}$ in the above expression we get,

$$S_{NG} = 2\sqrt{4\pi\lambda T} \int_{\epsilon}^{z_{max}} dz \frac{z_{max}^2}{z^2 \sqrt{z_{max}^4 - z^4}}$$

In order to get the renormalized expectation value of the Wilson loop, we must subtract the area of a constant cross-section surface that ends on the boundary configuration shown in Fig. 7(a). Hence we must subtract the sum of the area of two flat surfaces ending on the two straight lines from S_{NG} . Hence, the renormalized expectation value of the Wilson loop is given by,

$$\ln\langle W[C = | \cdot] \rangle_{\text{ren}} = S_{NG} - \sqrt{4\pi\lambda T} \left(2 \int_{\epsilon}^{z_{max}} \frac{dz}{z^2} \right)$$

The above expression can be simplified to yield,

$$\frac{\ln\langle W[C = | \cdot] \rangle_{\text{ren}}}{\sqrt{4\pi\lambda T}} = -\frac{2}{z_{max}} \int_{\epsilon/z_{max}}^1 \frac{dy}{y^2} \left(\frac{1}{\sqrt{1-y^4}} - 1 \right)$$

Hence, the static quark-antiquark potential $V(r)$ is given by,

$$V(r) = -\frac{\sqrt{\lambda}}{r} \left(\frac{4\pi^2}{[\Gamma(1/4)]^4} \right)$$

The $1/r$ dependence (in 3+1 dimension) of this potential is determined by scale invariance. Note that in the weak coupling limit, the potential is given by,

$$V(r) = -\text{const} \frac{\lambda}{r}$$

An important lesson that we learn from this calculation is that string description does not imply confining behavior. If the separation between the quark is small, then the string cannot stretch far into the bulk. Whereas if the separation is large then the string can stretch far into the bulk where the warp factor is bigger and the renormalized area is smaller. This explains why the potential is smaller when the separation is larger. In the following we shall briefly discuss other related calculations.

2.4 Wilson loop with a cusp

This Wilson loop is a pair of semi-infinite straight lines intersecting at an angle Ω (shown in Fig. 8). This can be used to calculate a quantity called cusp anomalous dimension. Calculations in QFT (resummation of rainbow graphs) show that this quantity has additional logarithmic divergence proportional to the opening angle Ω near the neighbourhood of the cusp. That is,

$$\ln\langle W(\text{cusp}) \rangle \propto \Omega \ln(L\Lambda)$$

where, L is the length of the loop and Λ is the UV cut-off. This quantity can be computed using the method described in this lecture. It can be shown that the renormalized area of the worldsheet ending on this cusp, has an extra logarithmic divergence ($\ln L/\epsilon$).

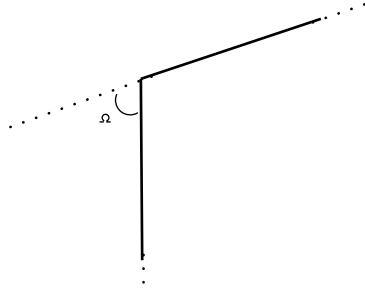


Figure 8: (a) Wilson Loop with a cusp

2.5 Correlators of Wilson loop

Other quantities of interest in QFT are the correlators of Wilson loops ($\langle W(\mathcal{C}_1)W(\mathcal{C}_2) \rangle, \dots$). These quantities can be expanded as an OPE in local operators. In order to compute these quantities using the *AdS/CFT* correspondence, we must extremize over all possible configurations of the worldsheet. For example,

$$\langle W(\mathcal{C}_1)W(\mathcal{C}_2) \rangle = \exp(\mathcal{F}_A, \mathcal{F}_B)$$

where \mathcal{F}_A and \mathcal{F}_B are the configurations shown in Fig. 9(a) and Fig. 9 (b). Let us now see

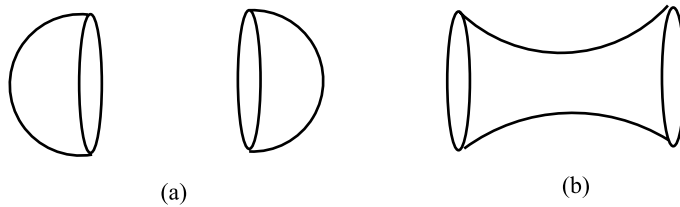


Figure 9: (a) Time Independent Boundary Configuration (b) Top-view of the configuration

how we can compute observables that are similar to the observables described above using the correspondence.

2.6 Monopole-Monopole Potential

This is the potential between two heavy monopoles and it can be computed by computing t'Hooft loop, which is the magnetic dual of Wilson loop. In *AdS*, this quantity (t'Hooft loop) can be computed by replacing the F-strings in example 2.3 by D-strings. We know (from problem set 1) that the end of a D string ending on D3 branes carry magnetic charge. More generally, we can compute t'Hooft loops by studying D-strings ending on t'Hooft loops. We must remember that the

tension of a D-string is different from that of a F-string.

$$\text{Tension of a D-string} = \frac{1}{g_s} \times \text{Tension of a F-string}$$

Hence, the monopole-monopole potential is given by,

$$V_{\text{mono}}(r) \sim \frac{\sqrt{\lambda}N}{r}$$

Similarly, we can compute quark-monopole potential.

2.7 k -surface observables

We can generalize the method used to compute Wilson loops to study observables associated with a k -dimensional surface specified on the boundary of AdS by computing the minimum area of the $k + 1$ -dimensional surface ending on it. Such observables have been studied by Graham and Witten in hep-th/9901021.

..... (to be continued)