

# 8.821 F2008 Lecture 19

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- Pointlike probes of the bulk - Geometric optics through D-branes.
- Baryons and Branes
  - Baryon vertices in  $\mathcal{N} = 4 SU(N)$  gauge theory
  - Giant gravitons
- Survey of other examples of the correspondence

## 1 Geometric Optics via D-Branes

Last time we observed that one could compute observables associated to  $k$ -dimensional submanifolds  $C \subset \partial\text{AdS}$  by studying  $\exp\left(-\underset{\partial\Sigma=C, \Sigma_{k+1} \subset \text{AdS}}{\text{Extremum}} \text{Vol}(\Sigma)\right)$  in analogy with Wilson lines.

The simplest example is the case  $k = 0$ , when  $C$  is a discrete point set. This would lead us to consider one-dimensional curves in AdS. This one dimensional curve is the worldline of some point particle excitation in AdS. Let  $C = \{-a, +a\}$  be the discrete point set and  $\Sigma$  denote a curve parametrized as  $z(\tau)$  with  $z(\tau = 0) = -a$  and  $z(\tau = 1) = +a$  as shown in Figure 1. The observable associated with  $C$  is clearly the two point function of some local operator  $\mathcal{O}$ . This local operator “couples” to a scalar field of mass  $m$  in the gravity theory. The usual prescription involves computing the two point function in the bulk and finding the on-shell action. Since we have a *free* scalar field, we can instead use a “first quantized” approach. The action for a point particle of mass  $m$  whose world-line is  $\Sigma$  is given by  $S[z] = m \int_{\Sigma} ds$ . The two point function for this particle  $\langle z(\tau = 0)z(\tau = 1) \rangle$  is given by the Feynman path integral  $Z[\pm a] = \int [dz(\tau)] \exp(-S[z]|_0^1)$ . In the limit of large  $m$ , we have

$$Z[\pm a] \sim \exp(-S[\underline{z}]) \tag{1}$$

where we have used the saddle point approximation.  $\underline{z}$  is the geodesic connecting the points  $\pm a$  on the boundary.

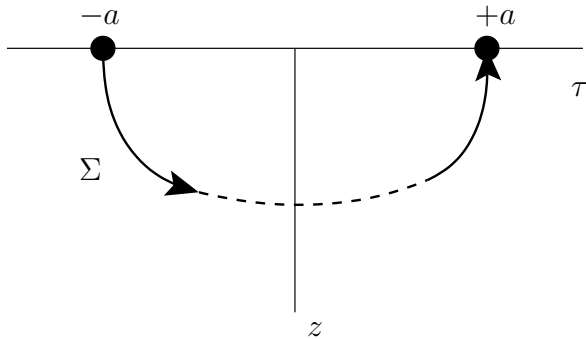


Figure 1: The curve  $\Sigma \subset \text{AdS}$  connecting the two points in  $C$ . The arrows on the curve indicate the orientation.

We now compute  $Z[\pm a]$  in the saddle point approximation. The metric restricted to  $\Sigma$  is given by  $ds^2|_{\Sigma} = \frac{L^2}{z^2}(1+z'^2)(d\tau)^2$ , where  $z' := dz/d\tau$ . This implies the action is

$$S[z] = \int d\tau \frac{L}{z} \sqrt{1+z'^2} \quad (2)$$

The geodesic can be computed easily by noting that the action  $S[z]$  does not depend on  $\tau$  explicitly. This implies we have a conserved quantity

$$h = z' \frac{\partial \mathcal{L}}{\partial z'} - \mathcal{L} = \frac{L}{z} \frac{1}{\sqrt{1+z'^2}} \quad (3)$$

$$\Rightarrow z'^2 = \left( \frac{L^2}{h^2} - z^2 \right) \frac{1}{z^2} \quad (4)$$

The above is a first order differential equation with solution  $\tau = \sqrt{z_{max}^2 - z^2}$ , with  $z(\tau = 0) = z_{max} = L/h = a$ . This is the equation of a semi-circle. Substituting the solution back into the action gives

$$S[\underline{z}] = \int d\tau \frac{L}{z} \sqrt{1+z'^2} = 2L \int_{\epsilon}^a \frac{dz}{\sqrt{a^2 - z^2}} \frac{a}{z} \quad (5)$$

$$= 2L \log \frac{2a}{\epsilon} + (\text{terms} \rightarrow 0 \text{ as } \epsilon \rightarrow 0) \quad (6)$$

Recall that the expectation value of a circular Wilson loop of radius  $a$  was independent of  $a$ . We argued that since we are computing this in a conformal field theory, the only scale in the problem is  $a$  and therefore the expectation value of the Wilson loop should be independent of  $a$ . This argument fails in the case at hand because there are two scales:  $a$  and  $\epsilon$ . The scale transformation is anomalous and this is manifested in the  $\epsilon$  dependence of  $S[\underline{z}]$ .

According to the GKPW prescription, the generating functional  $Z[\pm a]$  with the prescribed boundary conditions corresponds to the two point function  $\langle \mathcal{O}(+a) \bar{\mathcal{O}}(-a) \rangle$ . The saddle point approximation of large  $m$  corresponds to large  $\Delta$  in the CFT. Therefore

$$\langle \mathcal{O}(+a) \bar{\mathcal{O}}(-a) \rangle \sim Z[\pm a] \sim \exp(-S[\underline{z}]) \sim \frac{1}{a^{2mL}} \quad (7)$$

This is exactly what we expect for the two point function, since  $m^2 L^2 = \Delta(\Delta - D) \approx \Delta^2$  in the large  $\Delta$  limit. Without this anomaly, the two point function of the operator  $\mathcal{O}$  would be independent of

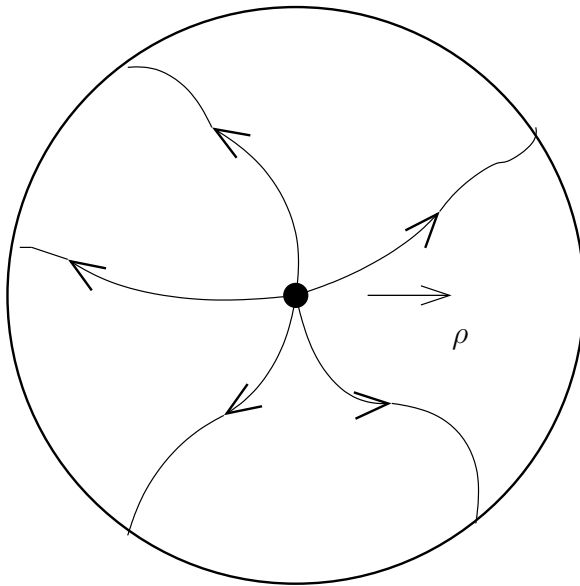


Figure 2:  $N$  F-strings ending on a D5 brane at the origin  $\rho = 0$  in global coordinates of  $\text{AdS}_5$ . The orientation of the strings is indicated with an arrow.

$a$  forcing  $\Delta = 0$ , which is impossible in a unitary CFT. This is similar to the anomaly that gives a non-zero scaling dimension of  $k^2/4$  to the operator  $\exp(ikX)$  for the 2d free boson CFT which has classical scaling dimension 0. Graham and Witten [1] showed that such a scale anomaly exists for surface observables for any even  $k$ . This relation between correlators of large  $\Delta$  observables and geodesics in the bulk theory offers a probe of the bulk spacetime [2].

## 2 Baryons and branes

In the previous section, we considered a very heavy point particle excitation in the bulk theory. The question that naturally arises is - Who are these heavy objects? A good candidate would be the D0 brane which is extremely heavy at weak coupling, but in Type IIB there are no BPS D0 branes. This is because the IIB theory has only even form RR potentials. In the absence of a conserved charge, this object would decay into a puff of closed string states. The next best thing we could consider are D $q$  branes wrapping  $q$ -cycles of  $S^5$ . The only closed cycles on  $S^5$  are the 0-cycle and the 5-cycle. This leaves us with only one choice - the D5 brane wrapping the  $S^5$ . This would correspond to a point particle in  $\text{AdS}_5$ . The tension of the D5 brane is  $T_5 = 1/(g_s l_s^6)$ . Naively, one would expect that the wrapped D5 brane would have a mass  $T_5 \text{Vol}(S^5) \sim L^5/(g_s l_s^6) \sim N\lambda^{3/2}/L$ . However, this turns out to be incorrect.

Recall (from Problem Set 1) that the action of the D5 brane includes the term

$$S_{D5} \supset \int_{M^6} F_{wv} \wedge C^{(4)} = - \int_{M^6} A_{wv} \wedge F^{(5)} \quad (8)$$

where  $M^6$  is the worldvolume of the D5 brane which happens to be  $S^5 \times \text{worldline}$  in  $\text{AdS}_5$ . The subscript  $wv$  indicates that the fields live on the worldvolume of the D5 brane. There are  $N$  units

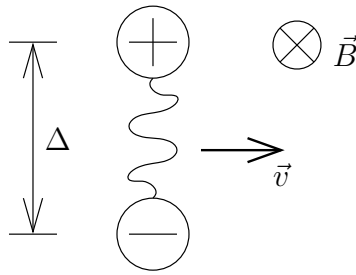


Figure 3: Electric dipole moving in a magnetic field.

of five form flux threading the  $S^5$  that stabilize its volume. This implies that

$$S_{D5} \supset -N \int_{\text{Worldline}} A_{wv} \quad (9)$$

The worldvolume gauge field has  $N$  negative point charge sources induced by the five-form flux. Since the space  $S^5$  is compact, this implies that we must have  $N$  positive point charge sources. Recall that the fundamental string that ends on a D-brane is electrically charged under the world-volume gauge field. Thus, we can cancel the net charge by attaching  $N$  F-strings to the D5 brane. Since these F-strings can't end anywhere, they extend all the way to infinity. Figure 2 illustrates this state in global coordinates.

The attached strings have a mass  $N/(\alpha') \times \int dz/z^2$  which is infinite and proportional to  $N$ . This object is not a dynamical point particle in IIB gravity theory on  $\text{AdS}_5$  since it is infinitely massive. It does not “couple” to a finite  $\Delta$  operator in the CFT. Recall from our discussion of Wilson loops that strings which extend all the way to the boundary of  $\text{AdS}$  correspond to infinitely massive quarks transforming in the fundamental of  $SU(N)$  in the CFT<sup>1</sup>. Thus, the D5 brane with  $N$  fundamental strings going off to infinity couples to a “baryon vertex” between  $N$  heavy, external quarks.

We can regulate the mass by considering a probe D3 brane at a finite radial position. The heavy quarks in the gauge theory have finite mass proportional to the radial position. This is completely analogous to what we did for the case of the Wilson loop. Now, we turn to perturbative string theory and compute the ground state for strings stretched between the D5 and D3 brane. The worldvolume of the D5 brane is  $S^5 \times \mathbb{R}$  and the worldvolume of the D3 brane is  $S^3 \times \mathbb{R}$ . Consider an open string stretching from the D5 brane (left end) to the D3 brane (right end). In quantizing the open string the boundary conditions for each coordinate are either Dirichlet-Dirichlet (DD), Neumann-Neumann (NN) or mixed (ND). The number of coordinates with ND boundary conditions is 8 for the 5-3 string. This implies that there is a fermionic massless ground state (See Polchinski II, section 13.4). This implies that the baryon vertex is antisymmetric under exchange of heavy quark fields.

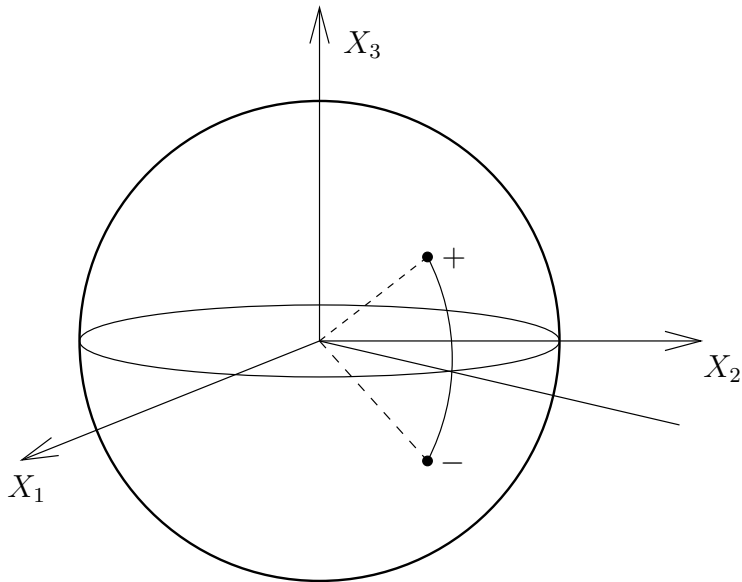


Figure 4: A D3 brane wrapping a 3-sphere inside a 5-sphere. Drawn above is the case of a 0-sphere  $X_3^2 = r^2$  at a polar angle  $\phi$ , embedded in a 2-sphere  $X_1^2 + X_2^2 + X_3^2 = L^2$ .

### 3 Giant Gravitons[3]

Consider an electric dipole consisting of charges  $+q$  and  $-q$  connected by a spring happily moving with a velocity  $v$  in a magnetic field  $B$  as shown in Figure 3. The forces acting on the dipole are the Lorentz force and the force exerted by the spring. The equation of motion is

$$m\ddot{\Delta} = -k\Delta + qvB \quad (10)$$

These forces oppose each other producing an equilibrium where the dipole has length  $\Delta = qvB/k$ . We can imagine the dipole moving on the surface of a two-sphere with a magnetic monopole at the origin producing the magnetic field.

Now, consider a D3 brane wrapping an  $S^3$  inside an  $S^5$ . We define the 5-sphere as the locus  $\sum_{i=1}^6 X_i^2 = L^2$ . A 3-sphere of radius  $r$  inside the  $S^5$  is specified by two equations -  $X_3^2 + X_4^2 + X_5^2 + X_6^2 = r^2$  and  $X_1 + iX_2 = \sqrt{L^2 - r^2}e^{i\phi}$ . This setup is shown in Figure 4 using a 0-sphere (two points) inside an  $S^2$ . The  $S^3$  is a contractible cycle in the  $S^5$  and since the D3 brane has a tension, the minimum energy configuration is a 3-sphere of radius zero. A spherical D3 brane is neutral under the RR 4-form. To see this consider two differential volume elements diametrically opposite on the 3-sphere. These two elements will trace out world-volumes with opposite orientations. Although each of these elements individually is charged under the 4-form, the total charge is zero. However, the D3 brane does have a dipole moment and if it moves in a “magnetic field”, the magnetic force would counter the force due to the tension and stabilize the volume of the  $S^3$ .

Consider the D3 brane shown in Figure 4 moving along the  $\phi$  direction with a velocity  $\dot{\phi}$ . To write down the DBI term, we first write the metric on the  $S^5$  considered as a two parameter family of

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<sup>1</sup>We take the gauge group as  $SU(N)$  and not  $U(N)$ . The  $U(1)$  factor in the CFT gauge group corresponds to the center of mass coordinate of the  $N$  D3 branes. For a discussion of this fact, see MAGOO page 58.

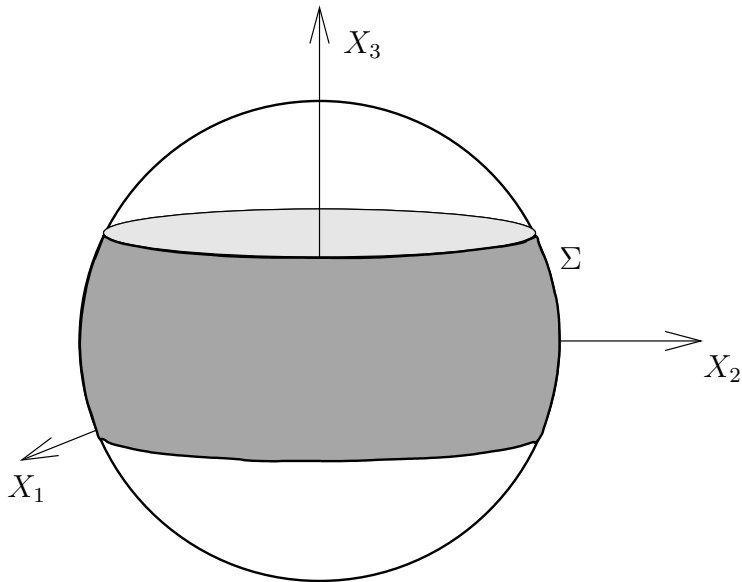


Figure 5: The worldvolume of the D3 brane is represented as the boundary of a four dimensional space  $\Sigma \times \text{time}$ .

$S^3$ 's as  $ds^2 = \frac{r^2}{L^2 - r^2} dr^2 + (L^2 - r^2) d\phi^2 + r^2 d\Omega_3^2$ . The D3 brane is located at fixed  $r$ , with the angular coordinate  $\phi$  varying as  $\dot{\phi}$ . Ignoring the world volume gauge field on the D3 brane, the DBI action is simply  $-T_{D3} \int_{WV} \sqrt{\det g}$ . On computing the induced metric (only the coordinate  $\phi$  varies with time), we have  $\det g = r^6 (1 - (L^2 - r^2) \dot{\phi}^2) \det(3\text{-sphere metric})$ . The DBI action can be written as

$$S_{DBI} = -T_{D3} \Omega_3 r^3 \int dt \sqrt{1 - (L^2 - r^2) \dot{\phi}^2} \quad (11)$$

The Aharanov-Bohm-Chern-Simons term can be written down in analogy with the case of an electric dipole (a 0-sphere) on a 2-sphere with a magnetic monopole at its center.  $\mathcal{L}_{ABCS} = -e \int dt v^i A_i(+)$   $+ e \int dt v^i A_i(-)$ . The  $+$  and  $-$  denote the positions of the positive and negative charges constituting the dipole. Let  $\Sigma$  be the two-dimensional manifold whose boundary is the worldline of the dipole as shown in Figure 5. If  $\dot{\phi}$  is the angular velocity of the dipole, it makes  $\dot{\phi}/(2\pi)$  revolutions per unit time. Then,  $\mathcal{L}_{ABCS} = -e \dot{\phi}/(2\pi) \int_{\partial\Sigma} A = -e \dot{\phi}/(2\pi) \int_{\Sigma} F = -eN \dot{\phi} r^2/L^2$  where  $N$  is the strength of the monopole charge. For the case of the D3 brane, let  $\Sigma$  be the four dimensional manifold whose boundary is one orbit of the D3 brane around the  $S^5$ . Then,

$$\mathcal{L}_{ABCS} = \frac{\dot{\phi}}{2\pi} \text{Vol}(\Sigma) \frac{B}{\text{Vol}(S^5)} \quad (12)$$

$$= \frac{\dot{\phi}}{2\pi} \left( \Omega_3 \int_0^r dr' r'^3 \right) \frac{2\pi N}{\Omega_5 L^5} \quad (13)$$

$$= \dot{\phi} N \frac{r^4}{L^4} \quad (14)$$

The angular momentum of the D3 brane is given by

$$l = \frac{\delta S}{\delta \dot{\phi}} = N \frac{r^4}{L^4} + \frac{m \dot{\phi} (R^2 - r^2)}{\sqrt{1 - (L^2 - r^2) \dot{\phi}^2}} \quad (15)$$

The effective potential for the D3 brane has the form  $V(r) \sim \frac{r^6}{lL^2} - Nr^4$  and has a minimum at  $r_*^2 = \frac{l}{N}l^2$ . The size of the D3 brane is determined by its angular momentum. In the SYM theory, the angular momentum maps to the R-charge. At a radius  $r_*$ , the energy of the D3 brane is  $E(r_*) = l/L$  which is the BPS bound for an operator with R-charge  $l$ . The analysis above used the DBI action for the D3 brane which is valid only when the radius of the D3 brane is large compared to  $\alpha'$  and this corresponds to  $l \sim L$ . Surprisingly, the D3 brane with angular momentum around the 5-sphere has the same quantum numbers as the KK mode of the graviton. In fact there is yet another BPS object with the same quantum numbers[5],[4]. This is a D3 brane wrapping a spatial  $S^3$  in  $AdS_5$  located at a point in the  $S^5$ . In the SYM theory there is a single BPS operator with R-charge  $l$ , the single trace operator  $\text{Tr} [X^l]$ . Thus we have three BPS objects on the gravity side with the same quantum numbers as the KK graviton. Since we are in the large  $\lambda$  regime, one can imagine that one linear combination of these BPS short multiplets remains short, while the other two combine to form a long multiplet. Observe that the largest radius the  $S^3$  can have is  $L$  which corresponds to a maximum angular momentum of  $l_{max} = N$ . This surprising fact called the stringy exclusion principle matches with the result that there is no single-particle BPS state in  $\mathcal{N} = 4$  with  $l > N$ . The operator  $\mathcal{O} = \det X$  has the maximum allowed  $l = N$ .

## 4 Other examples of the correspondence

There are other vacua of Type IIB string theory with an  $AdS_5$  factor. These would correspond to other four dimensional CFTs. For example, the space  $AdS_5 \times X^5$  satisfies the equations of motion if  $X^5$  is a five dimensional Einstein manifold. An Einstein manifold is a manifold whose Ricci curvature is proportional to the metric i.e.  $R_{MN} = \lambda g_{MN}$  with  $\lambda$  constant.

$\mathcal{N} = 4$  SYM is the low energy theory living on D3 branes in *flat space*. The transverse space to  $\mathbb{R}^{3,1} \subset \mathbb{R}^{9,1}$  is  $\mathbb{R}^6$  which is a cone over  $S^5$  (the celestial sphere). In fact any smooth space looks like  $\mathbb{R}^6$  locally and the low energy theory on the D3 branes only probes a neighbourhood of the geometry. So if we start with D3 branes at a point on a smooth space and look at the near horizon limit, the geometry would again look like  $AdS_5 \times S^5$ . One way to get around this is to consider D3 branes at a singular point in the geometry where the neighbourhood is no longer  $\mathbb{R}^6$ .

- $\mathbb{R}^6/\Gamma$  orbifolds: Orbifolds are manifolds except for a finite set of points which have a neighbourhood homeomorphic to  $\mathbb{R}^6/\Gamma$  where  $\Gamma$  is an element of a discrete subgroup of  $SO(6)$ . The near horizon limit at large  $\lambda$  gives a geometry  $AdS_5 \times S^5/\Gamma$ . There are prescriptions to find the resulting QFT. The basic idea is that we have a projection  $\Gamma$  and we have to decide how  $\Gamma$  acts on the branes. After that we just project onto the  $\Gamma$  invariant states.
- Cone over  $X_5$ : Not all singularities are orbifolds. A classic example is the conifold.

These spaces can have  $q$ -cycles and therefore one has static baryon-like excitations of finite energy. For the case of  $AdS_5 \times S^5$ , we found an infinite mass baryon-like excitation (D5 brane with  $N$  fundamental strings). It is also possible to deform these theories (by adding “fractional branes”) so that the gauge coupling runs. In the case of the  $\mathcal{N} = 4$  theory we have a line of fixed points and the dilaton is free.

## References

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