# 8.821 F2008 Lecture 23: Black Hole Thermodynamics

Lecturer: McGreevy   Scribe: Tongyan Lin

December 2, 2008

In today's lecture we'll discuss the laws of black hole thermodynamics and how AdS black holes are related to finite temperature CFTs, and Koushik will give a related presentation.

## 1   Laws of Thermodynamics

Recall from last time that for a black hole

$$
\begin{aligned}
Area &\sim Entropy \\
\kappa &\sim T
\end{aligned}
\tag{1}
$$

where $\kappa$ was the surface gravity. The near-horizon metric is

$$ds^2 \sim -\kappa^2 \rho^2 dt^2 + d\rho^2 + ... = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + ... \tag{2}$$

when we go to Euclidean time $\tau \equiv it$. If $\tau$ has periodicity $\tau \sim \tau + 2\pi/\kappa$ then the euclidean geometry is regular.

Recall the canonical ensemble thermal partition function is

$$Z_{th} = \mathrm{tr}\, e^{-H/T} \tag{3}$$

where $e^{-H/T}$ propagates the system with imaginary time $t = 1/iT$. Thermal equilibrium is equivalent to periodic euclidean time with period $1/T$, so we identify $\kappa$ with temperature $T$.

The laws of (stationary) black hole thermodynamics, analogous to the usual laws of thermodynamics, are:

- **0th (thermal equilibrium)**: $\kappa$ is constant over the event horizon. This means temperature is constant in space and time. Thus stationary black holes are in thermal equilibrium with constant temperature. John thinks the proof of the 0th law doesn't depend on the shape of the black hole, as long as its a stationary solution.

- **1st (conservation of energy)**:

$$dE = dM = \Omega dJ + \Phi dQ + \frac{\kappa}{8\pi G}dA \ (+PdV) \tag{4}$$

$\Omega dJ$ is the change in rotational energy, $\Phi dQ$ is the electrical energy, and $\frac{\kappa}{8\pi G}dA = TdS$ is heat exchange. This law relates the change in the energy (or equivalently mass) to changes in various properties of the black hole.

The last term describing mechanical work $PdV$ isn't present for black holes but IS for black branes...

- **2nd (entropy increases)**: This is the area theorem for a black hole we proved last lecture, $\dot{A} \geq 0$, since $S = \frac{A}{4\hbar G}$. (Proof of the exact relation between $S$ and $A$ in a later lecture.)

- **3rd (absolute zero entropy)**: $\kappa$ (or rather $T$) cannot taken to zero in a finite number of steps. This doesn't mean that $S(T = 0) = 0$, but it does probably mean at $T = 0$ there is a minimum in entropy.

These laws follow from Einstein's Equation, the energy condition we discussed last class, and assuming we have stationary black holes.

## 1.1  3rd law

Since we discussed the 2nd law last time, and the 0th and 1st laws are pretty convincing, we now provide some evidence for the validity of the 3rd law.

First, why isn't it true that $S(T = 0) = 0$? Counterexamples are everywhere if you just open your eyes to them:

- It is well known to some people that there exist supersymmetric theories with LARGE ground state degeneracies, $\propto e^{Q^{\alpha}}$ where $Q$ is the charge and $\alpha$ is some power. So, $S(T = 0) = \ln(\text{degeneracies}) \sim Q^{\alpha}$.

- The Kerr-Newman black hole is another counterexample. Here are some facts about the KN black hole that you can easily derive or look up:

$$\begin{aligned} A &= 4\pi(2M(M + \mu) - Q^2) \\ \mu &= \sqrt{M^2 - Q^2 - J^2/M^2} \\ \kappa &= 4\pi\mu/A \end{aligned} \tag{5}$$

The black hole is extremal when $\mu = 0$. (This is also the BPS bound when the black hole is supersymmetric.) If $\mu < 0$ then there is a naked singularity. Note that that when $\mu = 0$, $\kappa \propto T = 0$ but $S \propto A \neq 0$.

As for the claims of the 3rd law, we have some anecdotal evidence. Let's consider a non-extremal KN black hole with $J = 0$, in other words a non-extremal RN black hole, so $Q < M$. (Everywhere $Q$ is really $|Q|$).

What can we do to try to make this black hole extremal? We need to throw on some charge $q$ and mass $m$, such that the black hole becomes extremal, namely

$$M + m = Q + q \tag{6}$$

How, the mass $m$ is attracted to the beautiful black hole by a force $F \sim Mm/r^2$ but the charge $q$ is repulsed by a force $F \sim Qq/r^2$. Thus for the matter to fall in freely, $Mm > Qq$. According to some mysterious algebra, this relation along with $Q < M$ actually implies $Q + q < M + m$. Therefore you have to force the matter onto the black hole, which somehow adds heat and prevents you from cooling the black hole. Or you have to throw in infinitesimal little bits which takes FOREVER.

# 2 CFT at finite temperature

We're going to use the power of AdSCFT to describe CFTs at finite temperature with black holes.

In particular we mean a $3 + 1$ dimensional relativistic CFT. The partition function is

$$Z(\tau) = \mathrm{tr} e^{-H/T} = e^{-F/T} \tag{7}$$

with free energy $F$, on a space with geometry $S^1_{th} \times \Sigma_3$ where the $S^1$ has radius $1/T, \tau \sim \tau + 1/T$ and $\Sigma_3$ is some 3 manifold. We can give $\Sigma_3$ finite volume as an IR regulator.

This is a deformation of the IR physics (modes with $\omega \gg T = E_{KK}$ don't notice).

For large $V_3 = Vol(\Sigma_3)$, then $F = cV_3 T^4$ which is clear from extensivity of $F$ and dimensional analysis.

# 3 AdS black holes

This object goes by many names, such as planar black hole, Poincare black hole, black brane... This is a black hole in $AdS_{D+1}$, but probably many of the equations below mean $D = 4$. The metric is

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2}\left(-fdt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right) \\ f &= 1 - \frac{z^4}{z_m^4} \end{aligned} \tag{8}$$

We again put the $\vec{x}$ coordinates on a finite volume space, for example in box of volume $V_3, x \sim x + V_3^{1/3}$, periodic BCs. Notice that if $f = 1$ we get the Poincare AdS metric, and in fact $f$ only deviates from 1 at larger $z$ representing the fact that this is an IR deformation.

Whence:

- It solves Einstein's equations with a cosmological constant $\Lambda = \frac{(D+1)(D+2)}{2L^2}$ and asymptotes to Poincare AdS, differing only in the IR region with a horizon at $z = z_m$, fixed $t$.

- It's the double Wick rotation of the confining solution with $t = iy_{there}, y = -it_{there}$.

- Analogous to how we got the AdS solution from the near-horizon limit of D3-branes, it's the near-horizon limit of black 3-branes in $\mathbb{R}^{9,1}$, in particular the near-extremal RR soliton with geometry:

$$
\begin{aligned}
ds^2 &= \frac{-f dt^2 + d\vec{x}^2}{\sqrt{H(r)}} + \sqrt{H(r)}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right) \\
H(r) &= 1 + \frac{\tilde{L}^4}{r^4} \\
f(r) &= 1 - \frac{r_H^4}{r^4}
\end{aligned}
\tag{9}
$$

(Again $f = 1$ gives the usual RR soliton.) Note that there also exists a black hole which asymptotes to GLOBAL AdS (with boundary $S^1 \times S^3$), which is known as AdS-Schwarzchild which describes a CFT on $S^3$ at finite temperature T.

Let's check out the horizon properties so we can find the usual thermodynamic quantities we're interested in. The near-horizon metric is

$$
ds^2 \sim \kappa^2 \rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_m^2} d\vec{x}^2
\tag{10}
$$

where $\kappa = 2/z_m$ and the temperature is $T = \kappa/(2\pi) = 1/(\pi z_m)$. Meanwhile the area of horizon is

$$
A = \int_{z=z_m, \text{fixed} t} \sqrt{g} d^3 x = \left(\frac{L}{z_m}\right)^3 V_3
\tag{11}
$$

Therefore the entropy is (in the deconfined phase of the gauge theory):

$$
S = \frac{A}{4G_5} = \frac{L^3}{4G_5}\frac{V_3}{z_m^3} = \frac{N^2}{2\pi}(\pi T)^3 V_3 = \frac{\pi^2}{2} N^2 V_3 T^3
\tag{12}
$$

(Recall from long ago that $\frac{L^3}{4G_5} = \frac{N^2}{2\pi}$.)

Again, we want some anecdotal evidence (at least) to support the claim that this describes a CFT in thermal equilibrium. There are some checks. One is just by checking the first law, which relates horizon quantities such as $T, S$ to global quantities such as the free energy $F$. Koushik will explain just how in his talk, as sketched below.

Consider again the partition function which we claim is $Z_{CFT} = e^{-S_g} = e^{-\beta F}$. $S_g$ is the onshell gravity action for the black hole solution and is equal to

$$
S_g = S_{EH} + S_{GH} \ (+S_{ct}).
\tag{13}
$$

$EH$ means Einstein-Hilbert and $GH$ means Gibbons Hawking, $S_{GH} \sim \int_{\partial M} d^D x \sqrt{\gamma} K$ is a boundary term for the action such that when we vary $S_g$ we get the usual equations of motion. $ct$ means

counter-term, as we need to subtract some divergences as $r \to 0$. You can see hep-th/9902121 for some details.

Thus by plugging in the AdS planar black hole solution we obtain the free energy, and consistent with the entropy calculation we did above we obtain

$$\frac{F}{V} = \frac{L^2}{16\pi G_5}\frac{1}{r_H^4} = \frac{\pi^2}{8}N^2 T^4 \tag{14}$$

$V = V_3$ probably. Also, $z_m = r_H$. This we claim is the strong-coupling free energy. As for the field theory calculation it has been done at weak coupling and one obtains $4/3$ times the answer at strong coupling.

Let's consider also the boundary stress tensor $T^{\mu\nu}$ which couples to the induced metric on the boundary $\gamma_{\mu\nu}$. The energy is $E = V\sqrt{\gamma}T_t^t$ and the pressure $P$ involves similar expressions with $x, y, z$ components of $T$. This is also a straightforward calculation using the action above and one obtains $(E/V) = 3/(2r_H^4)$ and $P = 1/(2r_H^4)$ which satisfies $E = 3P$, so $T_\mu^\mu = 0$ as required in a CFT. One can do the same for the AdS Schwarzchild black hole, where $E = 3/(2r_H^4) + 3L^2/(8G)$. As $T \to 0, r_H \to \infty$ but $E$ is nonzero and in fact matches the calculation for the zero-point (Casimir) energy for $\mathcal{N} = 4$ SYM on a sphere of radius $L$. (There is no extra $4/3$ factor because...)