8.821 F2008 Lecture 24

Blackhole Thermodynamics (Con'd)

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1 Classical Blackhole Thermodynamics (Con'd)

Previously we have stated the third law for blackhole thermodynamics. To rephrase the result:

3rd Law of blackhole thermodynamics. *Extremal blackholes are unnatural.*

In spite of this, extremal blackholes are beloved by string theorists, because the computations are easier. In particular, $S_{\text{extr. BH}}(T = 0) = \ln(\#\text{ of ground states})$, and SUSY implies that this is independent of couplings and hence can be computed in the weak coupling limit.

Previously we have also stated the second law, but in which we only considered blackholes alone. We now consider a more general version the second law in which both blackholes and normal stuff are included:

Generalized 2rd Law (GSL) of blackhole thermodynamics. *[Bekenstein PRD **9** 3292 (1974)]*

$$S_{\text{total}} = S_{\text{normal stuff}} + \zeta \frac{A}{\hbar G_N}$$
$$\delta S_{\text{total}} \geq 0 \quad \text{for physical processes.}$$

where ζ is a number yet to be determined.

The idea behind the GSL is that it will be possible to build perpetual machines if it is false. However, there is a problem with the "lowering-the-box" energy extraction scheme described in a previous lecture. To be more specific, suppose we lower an (infinitesimal) box of entropy from ∞ to the event horizon and extract energy along the way as described in the previous lecture. When the box is at the event horizon, we know that $E_{\text{box}}(r = r_s) = 0$, and if we let go at that point, $\Delta A = 0$ for the blackhole according to the first law. Hence the GSL is violated.

The other laws of classical blackhole thermodynamics also contain "flaws":

- *Flaw # 1.* Classically a blackhole cannot radiate, so $T_{BH} = 0$?
- *Flaw # 3.* Classically, the area of *each* individual blackhole must be non-decreasing. But thermodynamics should require only that $\delta S_{\text{total}} \geq 0$. Thus the blackhole thermodynamics and ordinary thermodynamics are not in good parallel.[1]

The apparent contradictions in the blackhole thermodynamic laws can be resolved by realizing that the blackhole entropy $S_{\text{Sek}} = A/\hbar G_N$ contains a factor of $\hbar$, hence $S_{\text{Bek}} \to \infty$ in the classical limit ($\hbar \to 0$) when A and G_N are fixed. In particular, for the box-lowering scenario, even though the change in area δA is infinitesimal, it can still lead to a finite change ΔS in entropy. Hence there is no contradiction with the GSL.

Similarly, from the 1st Law $dM = T_{\text{Bek}} dS_{\text{Bek}}$. As $\hbar \to 0$, $T_{\text{Bek}} = \hbar\kappa/8\pi\zeta \to 0$. Hence flaw # 1 does not lead to a contradiction in the classical limit.

Also, as $\hbar \to 0$, *any* finite δA for any blackhole will lead to $\Delta S = \infty$, so flaw # 3 does not lead to a contradiction in the classical limit.

2 Quantum Blackhole Thermodynamics

Our major result is the Hawking radiation law:

Hawking Radiation. *A blackhole radiates like a blackbody with* $T = T_{BH} = \dfrac{\hbar\kappa}{2\pi}$.

Let's first consider some consequences before considering the explanation:

1. This fixes the numerical constant ζ to be 1/4. i.e., $S_{\text{BH}} = \dfrac{A}{4\hbar G_N}$.
2. The area of individual blackhole can decrease, thus fixes flaw # 3 of GSL.
3. The resulting dA satisfies GSL for reasonable equation of state (E.O.S.). For example, consider radiation in $4d$. We have $\mathcal{E} = (1/4)aT^4$ and $S = (1/3)aT^3$. Treating blackhole as a blackbody in flat space, $dS_{\text{rad}} = \frac{4}{3}\frac{dE}{T_{\text{BH}}}$, while energy conservation implies that $dM = -dE$ and hence $dS_{\text{BH}} = -\frac{dE}{T_{\text{BH}}}$. Summing up[2], $dS_{\text{total}} = d(S_{\text{rad}} + S_{\text{BH}}) = \frac{1}{3}\frac{dE}{T_{\text{BH}}} > 0$.
4. Let's return to the box-lowering puzzle,
   (a) Bekenstein bound (Conjecture). Entropy in a box of linear size R must satisfy $S_{\text{box}} \leq 2\pi ER$.
   (b) *Buoyancy force.* Consider a box near event horizon, so that it's top side is at r_2 and the bottom side is at r_1 (see Fig. 1). The local temperature T_1 (T_2) on the top (bottom)

[1] Perhaps this is not a big deal, since individual blackholes can merge.
[2] Consequently, this says that the speed of sound is $c^2_{\text{sound}} = 1/3$.

side satisfies $T_2/T_1 = \sqrt{g_{tt}(r_1)}/\sqrt{g_{tt}(r_2)}$. Hence $T_1 > T_2$. Consequently, there is more radiation from the bottom than from the top[3], which produces a buoyancy force $\propto$ displaced radiation.
As a result, there is a critical $r_{\text{crit}} > r_s$ for which $E(\text{displaced radiation}) = E_{\text{box}}(r_{\text{crit}})$. By dropping the box at r_{crit}, $\Delta S_{\text{BH}} = E/T_{\text{BH}}$, but $S_{\text{box}} \leq S_{\text{displaced radiation}} = E/T_{\text{BH}}$

Now let's return to the question of why T_{BH} takes this particular form. For a generic (possibly non-extremal) blackhole, the event horizon is located at a zero of the emblackening factor f.

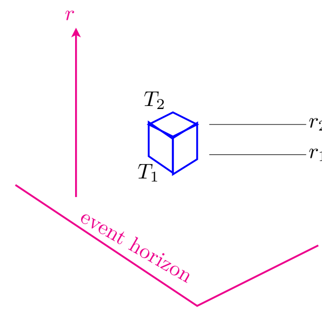

Figure 1: Buoyancy force on a box.

For example, for Schwarzchild blackhole,

$$\begin{aligned} ds^2_{\text{Sch}} &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\ &\approx \kappa^2 R^2 dt^2 + dR^2 + r(R)^2 d\Omega^2 \\ &= -R^2 d\eta^2 + dR^2 + \dots \\ &\approx -dT^2 + dZ^2 + dX^2 + dY^2 \end{aligned}$$

where in the above we have used:[4] $f = 1 - r_H/r$, $r_H = 2GM$, $R(r) = \sqrt{r(r-2GM)} + \dots$, $\kappa = 1/4GM$, $\eta = \kappa t$, $T = R\sinh\eta$ and $Z = R\cosh\eta$.

Notice that after the transformation, we obtain the Minkowski space $\mathbb{R}^{3,1}$. In other words, the Rindler space is a Minkowski space in the coordinate frame of a uniformly accelerating observer.

Graphically, what we get is Fig. 2. Notice that the two lines defined by $\{r = 2GM, \eta = \infty\}$ and $\{r = 2GM, \eta = -\infty\}$ divide the space into four regions { I, II, III, IV }.

The key observation is that region I is self-contained. i.e., region II and III can't communicate with it, while information from region IV passes through the line $\{r = 2GM, \eta = -\infty\}$ and hence corresponds to *initial data*.

Therefore, at $T = 0$, the $Z < 0$ (marked by L on the figure) degree of freedom (d.o.f.) are totally irrelevant and useless for region I. Thus we should trace over them when computing stuff in region I.

Now, Claim:

$$\rho_R = \text{tr}_L |\text{g.s.}\rangle\langle\text{g.s.}| = \frac{1}{\mathcal{Z}} e^{-2\pi H_R}$$

where H_R is the riddler Hamiltonian:[5]

$$H_R = \text{“}\frac{\partial}{\partial\eta}\text{”} = \int_{\text{const. } T \text{ surface}} T_{ab} \left(\frac{\partial}{\partial\eta}\right)^a n^b$$

and $\mathcal{Z} = \text{tr}_R\, e^{-2\pi H_R}$.

[3] analogous to more water at the bottom of sea
[4] η is called the "Rindler time" and plays the role of rapidity
[5] Note that $[H_R] = 1$ since the riddler time $[\eta] = 1$.

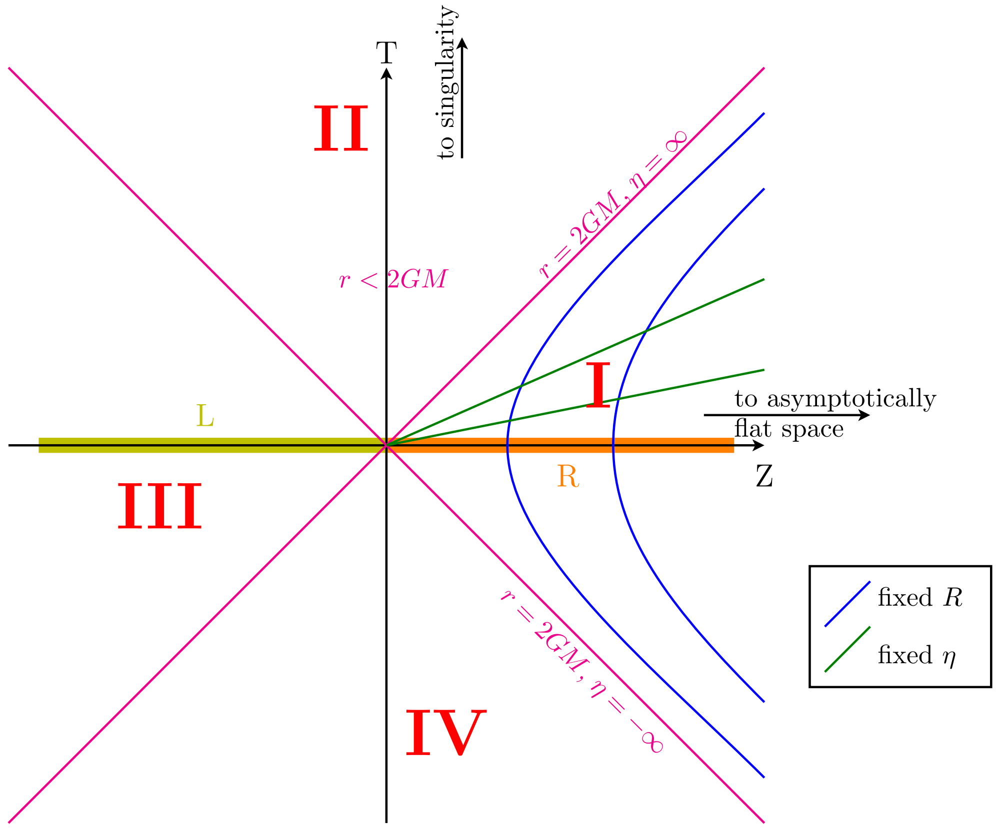

Figure 2: The geometry of the Rindler space.

Notice that the density matrix we obtained is *not a pure state* but is entangled with the L d.o.f. (hence $S = -\mathrm{tr}\ \rho_R \ln \rho_R \neq 0$). This is a thermal density matrix with temperature $T_{\text{riddler}} = 1/2\pi$.

Proof of Claim [Unruh 1976]. Consider a scalar field ϕ in Rindler space. A complete set of commuting d.o.f's at $T = 0$ is

$$\phi(x, y, z) = \begin{cases} \phi_R(x, y, z) & \text{for } Z > 0 \\ \phi_L(x, y, z) & \text{for } Z < 0 \end{cases}$$

ϕ_R and ϕ_L commute because they are at spacelike separated points. Any wavefunctional of the field ϕ can then be written as $\Psi = \Psi[\phi_L, \phi_R] = \langle \phi_L \phi_R | \Psi \rangle$.

The ground state is just the Minkowski space vacuum. The Feynman-Kac formula gives us a path integral representation of its wavefunctional:

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{x^0 > 0, \phi(\vec{x}, x^0 = 0) = (\phi_L, \phi_R)} [d\phi] e^{-S_{\text{Eucl}}[\phi]}$$

This equation is written using constant-$x^0 = -iT$ time slices. We can gain some insight by instead slicing up the path integral by constant euclidean Rindler time $\theta (= -i\eta)$ (see Fig. 3(a) for

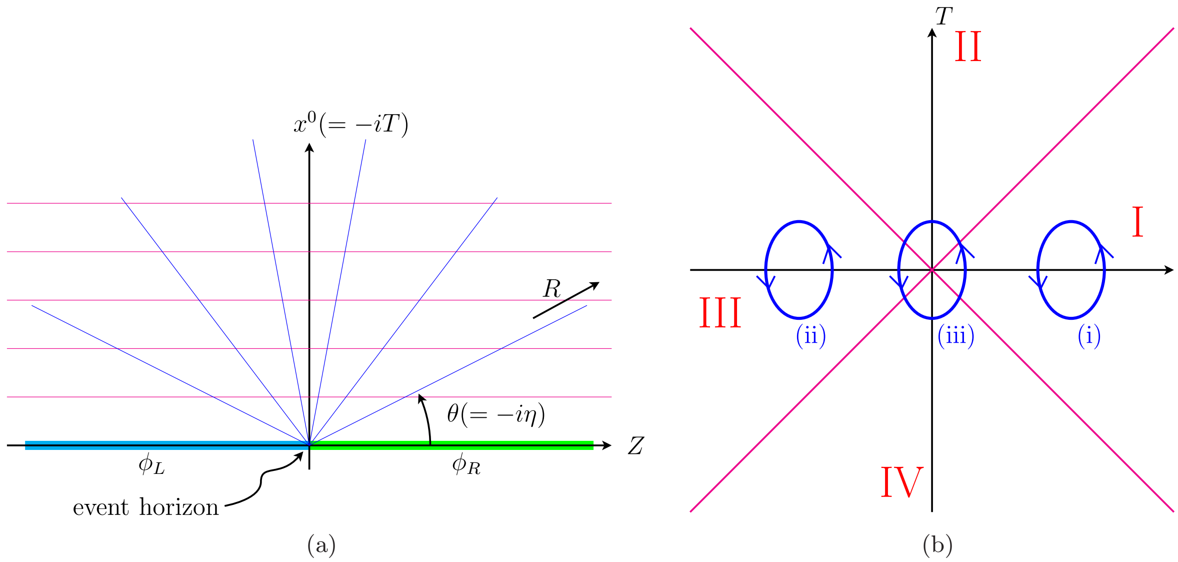

Figure 3: (a) Change in slicing sequence. The magenta lines are the old x^0 slices while the blue lines are the new θ slices. (b) Fluctuation in different regions of the Rindler space.

illustration); euclidean Rindler time is just the angular coordinate in the x^0, Z plane. This yields:[6]

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{\substack{0<\theta<\pi \\ \phi(\theta=0)=\phi_R \\ \phi(\theta=\pi)=\phi_L \\ \partial_R\phi(R=0)=0}} [d\phi] e^{-S_{\text{Eucl}}[\phi]} = \frac{1}{\sqrt{\mathcal{Z}}} \langle \phi_L | e^{-\pi H_R} | \phi_R \rangle \tag{1}$$

The last condition $\partial_R\phi(R=0) = 0$ is simply a requirement that the function ϕ is regular at the origin. We identify the RHS of (1) $\langle \phi_L | e^{-\pi H_R} | \phi_R \rangle$ as the transition amplitude.

Hence,

$$\begin{aligned} \langle \phi_R | \rho_R | \phi'_R \rangle &= \int [d\phi_L] \Psi^*[\phi_L, \phi_R] \Psi[\phi_L, \phi'_R] \\ &= \left(\frac{1}{\sqrt{\mathcal{Z}}} \right)^2 \int [d\phi_L] \langle \phi_R | e^{-\pi H_R} | \phi_L \rangle \langle \phi_L | e^{-\pi H_R} | \phi'_R \rangle \\ &= \frac{1}{\mathcal{Z}} \langle \phi_R | e^{-2\pi H_R} | \phi'_R \rangle \end{aligned}$$

■

Some results and comments:

1. a constantly accelerated observer in $\mathbb{R}^{n,1}$ sees a *Unruh radiation* with[7] $T_{\text{proper}} = \dfrac{R}{2\pi} = \dfrac{a}{2\pi}$.
2. By the equivalance principle, the same happens to a blackhole. The corresponding Schwarzchild time is $t = \eta/\kappa$, which implies that $T_{\text{BH}} = T_{\text{riddler}} = \dfrac{\kappa}{2\pi}$.

[6] We are transforming the variable that the function takes, rather than the function itself, and it is the function that's being integrated. Thus there is no Jacobian involved here.

[7] here the proper time is defined by $ds^2_{\text{proper}} = -R^2 d\eta^2$

3. Whence does the radiation comes from? There are three types of fluctuations (see Fig. 3(b)):
   (i) is just ordinary pair production fluctuation, with lifetime $\propto \hbar/E$
   (ii) is something that we need not care in region I
   (iii) corresponds to stuff that enters at $\eta = -\infty, R = 0$ and falls back at $\eta = +\infty, R = 0$. In Schwarzchild time this particle stays forever. Hence this particle is real.

4. What we have calculated is the density matrix of a quantum field at $r \sim r_H$. Not all energy will get to ∞. Thus we have to include a *greybody factor*, so that:

$$N_p^{\binom{b}{f}} = \frac{\Gamma_p}{e^{\hbar\omega/T_H} \mp 1}$$

   where Γ_p is the absolute cutoff for mode p.

5. *Information paradox.* Blackhole can form from dictionary and then evaporate into thermal radiation. The thermal density matrix is determined by one number only.
   Q1. Is this a unitarity evolution?
   Q2. Entropy results from coarse-graining. What are the microstate?
   Q3. How is the information stored in the blackhole?
   via AdS/CFT it is possible to answer Q1 and Q2.