1. **Branes ending on branes.**

The D$p$-brane effective action contains a term of the form

\[ S \ni \int_{D_p} F \wedge C_{p-1}, \]

where \( C_{p-1} \) is the RR \( p-1 \) form, which couples minimally to D\((p-2)\)-branes. Show that a D\((p-2)\) brane can end on a D\(p\) brane without violating the Gauss law for the RR fields involved. Interpret the boundary of the D\((p-2)\)-brane in terms of the worldvolume theory of the D\(p\) brane. (If you like, focus on the case \( p = 3 \).)

2. **Timelike oscillators are evil.**

Show that the commutation relation \([a,a^\dagger] = -1\) (which we found for the oscillators made from the time coordinates of the string) implies that either

a) the energy \( H = -a^\dagger a + E_0 \)

or

b) there are states with negative norms.

3. **Extremal Reissner-Nordstrom black hole.**

As a warmup for the 10-d RR soliton, let’s remind ourselves how the extremal RN black hole works.

a) Consider Einstein-Maxwell theory in four dimensions, with action

\[ S_{EM} = \frac{1}{16\pi G_N} \int d^4 x \sqrt{g} \left( R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \]

\(^1\)The \(-1\) comes from \( g^{00} = -1 \).
Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g_{\mu\nu}}$ implies that

$$R_{\mu\nu} = aG_N \left( 2F_\mu F_\nu - \frac{1}{2} g_{\mu\nu} F^2 \right)$$

for some constant $a$.

b) Consider the ansatz

$$ds^2 = H^{-2}(\rho) (-dt^2) + H^2(\rho) \left( d\rho^2 + \rho^2 d\Omega^2_2 \right),$$

$$F = b dt \wedge d \left( H(\rho)^{-1} \right)$$

where $b$ is some constant. Show that the Einstein equation $0 = \frac{\delta S_{EM}}{\delta g_{\mu\nu}}$ and Maxwell’s equation $0 = \frac{\delta S_{EM}}{\delta A_\mu}$ are solved by the ansatz if $H$ is a harmonic function on the $\mathbb{R}^3$ whose metric is

$$\gamma_{ab} dx^a dx^b := d\rho^2 + \rho^2 d\Omega^2_2.$$ 

Recall that $H$ is harmonic iff $0 = \Box H = \frac{1}{\sqrt{\gamma}} \partial_a ( \sqrt{\gamma} \gamma^{ab} \partial_b H )$.

c) Find the form of the harmonic function which gives a spherically symmetric solution; fix the two integration constants by demanding that i) the spacetime is asymptotically flat and ii) the black hole has charge $Q$, meaning $\int_{S^2 \text{ at fixed } \rho} *F = Q$.

d) Take the near-horizon limit. Show that the geometry is $AdS_2 \times S^2$. Determine the relationship between the size of the throat and the charge of the hole.

[If you get stuck on this problem, see Appendix F of Kiritsis’ book.]

d) If you’re feeling brave, add some magnetic charge to the black hole. You will need to change the form of the gauge field to

$$F = b dt \wedge d H(\rho) + G(\rho) \Omega_2$$

where $\Omega_2$ is the area 2-form on the sphere, and $G$ is some function.

4. RR soliton.

In this problem we’re going to check that the RR soliton is a solution of the equations of motion. The action for type IIB supergravity, when only the metric and the RR 5-form and possibly the dilaton are nontrivial can be written as

$$S_{IIB} = \frac{1}{16\pi G_N} \int d^{10} x \sqrt{g} \left( e^{-2\Phi} (\mathcal{R} + 4 \partial_\mu \Phi \partial^\mu \Phi) - \frac{1}{5!} F^5 \ldots F^5 \ldots + \ldots \right)$$

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(The self-duality constraint \( F^5 = \star F^5 \) must be imposed as a constraint, and means that \( dF^5 = 0 \) implies the equations of motion for \( F^5 \).) By the way, this is the action for the string frame metric.

a) Show that the equations of motion from this action imply

\[
\mathcal{R}_{\mu\nu} = a G_N e^{2\Phi} \left( 5 F^5_{\mu\nu\alpha} F^5_{\alpha} \cdots - \frac{1}{2} g_{\mu\nu} (F^5)^2 \right)
\]

for some constant \( a \).

b) Plug the following ansatz into the equations of motion:

\[
ds^2 = \frac{1}{\sqrt{H(r)}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{H(2)} dy^2
\]

\[
F = b(1 + \star) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dH^{-1}
\]

\[
\Phi = \phi_0
\]

\((b, \phi_0 \text{ are constants.})\) Determine the constant \( b \) and the condition on the function \( H \) for this to solve the equations of motion.

To do this, there are two options – some kind of symbolic algebra program like Mathematica or Maple, or index-shuffling by hand. The latter is much more easily done using ‘tetrad’ or ‘vielbein’ methods. I always forget these and have to relearn them every time. For a lightning review of the vielbein method of computing curvatures, I recommend d’Hoker-Freedman \textit{http://arXiv.org/pdf/hep-th/0201253}, pages 100-101, or Argurio \textit{http://arXiv.org/pdf/hep-th/9807171}, Appendix C. To help with the former option, I’ve posted an example curvature calculation in Mathematica on the pset webpage.

Note, by the way, that for values of \( p \) other than 3, the dilaton is not constant. With hindsight, this specialness of \( p = 3 \) is related to the fact that this is the critical dimension for YM theory, where \( g_{YM} \) is dimensionless.