MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2008

# Problem Set 4
*Fields in AdS*

**Due:** Tuesday, November 4, 2008.

1. **One more symmetry.**

   Define $x^A \equiv (z, x^\mu)$, so that the Poincaré patch metric takes the form

   $$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} = \frac{dx^A dx_A}{z^2}.$$

   Show that this metric is preserved by the following *inversion* symmetry:

   $$x^A \to \frac{x^A}{x^B x_B}$$

   *i.e.* $z \to \frac{z}{z^2+\vec{x}^2}, x^\mu \to \frac{x^\mu}{z^2+\vec{x}^2}$. This is a conformal transformation which is not continuously connected to the identity.

   [Bonus question: what does this do to the region of the global $AdS$ space covered by the Poincaré patch coordinates?]

2. **Schrödinger description of AdS instabilities.**[1]

   a) Consider the scalar wave equation in poincaré $AdS$, in momentum space:

   $$0 = (\Box - m^2)\phi,$$

   with $\phi(x,z) = e^{ik\cdot x}\phi(z)$ By redefining the independent variable, rewrite the scalar wave equation as a Schödinger equation:

   $$(-\partial_z^2 + V(z))\Psi(z) = \omega^2 \Psi(z)$$

   with $\omega^2$ playing the role of energy. (Hint: Rewrite $\Psi(z) = A(z)\phi(z)$, and choose the function $A$ to set to zero the term in the wave equation multiplying $\Psi'(z)$.)

[1] I learned about this trick from Sean Hartnoll. If you get stuck or want to see a recent application of this technique, see *e.g.* appendix A of 0810.1563.

b) A property of a field configuration which generally determines whether it is allowed to participate in the physics (*i.e.* whether it is fixed by boundary conditions, or whether it can happen on its own) is whether it is *normalizable.* In Euclidean space, this generally just means that the euclidean action evaluated on the configuration is finite, since then it contributes to the path integral with a finite Boltzmann factor $e^{-S[\phi]}$. In Minkowski signature, a field configuration should be considered normalizible if it has finite energy; for a scalar field, this energy is

$$\mathcal{E}[\phi] \equiv \int_\Sigma T_{AB}\xi^A n^B$$

where the integral is over some fixed-time slice $\Sigma$, $n$ is a unit normal vector to $\Sigma$, $\xi$ is a time-like killing vector, and $T_{AB}$ is the stress tensor for the bulk field, $T_{AB} \equiv \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{AB}}$.

Given the redefinition you found in part a), what is the relationship between normalizable wave functions in the usual QM sense (*i.e.* $||\Psi||^2 < \infty$) and normalizable solutions of the AdS wave equation (*i.e.* those with finite energy)?

c) Set $\vec{k} = 0$. Show that when $m^2$ passes through the value $-|m^2_{BF}|$ from above, this Schrödinger equation develops a single normalizable negative-energy state.[2] (Recall that the BF bound is $m^2 \geq m^2_{BF} = -\frac{D^2}{4L^2}$.) This corresponds by the map above to a normalizable mode with *imaginary* frequency – *i.e.* a linear instability.

3. **Dimensions of vector operators.**

By studying the boundary behavior of solutions to the bulk Proca equations, derived from the action

$$S_{\text{bulk}}[A] = -\kappa \int_{AdS} d^{p+2}x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu\right)$$

compute the scaling dimension $\Delta$ of the current coupled to a bulk vector field $A_\mu$ of arbitary mass:

$$S_{\text{bdy}} = \int_{\partial AdS} d^{p+1}x\sqrt{\gamma}A_\mu J^\mu.$$

For purposes of this problem, define $\Delta$ to be the power of $z$ of the *subleading* power-law solution of the bulk equation near the boundary at $z = 0$.

[2]If you want to learn more about potentials of this form, see H. Hammer and B. Swingle, Annals Phys. 321 (2006) 306-317 [arXiv:quant-ph/0503074v1].

4. **Saturating the unitarity bound.**[3]

For this problem we will work in Euclidean $AdS$. Feel free to redo the problem for the Lorenzian case (using the notion of normalizability described in problem two).

a) Consider the usual bulk action for a (free) scalar field in $AdS_{D+1}$:

$$S_{\text{usual}}[\phi] = -\frac{1}{2}\int d^D x dz \sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right)$$

Show that for $m^2L^2 > -\frac{D^2}{4} + 1$,

$$S_{\text{usual}}[z^{\Delta_-}] = \infty.$$

That is, the integral over the radial direction in this action diverges near $z = 0$ when evaluated on a solution of the form $\phi \sim z^{\Delta_-}$, where $\Delta_-$ is the *smaller* root of $\Delta(\Delta - D) = L^2m^2$; such a solution is non-normalizable. On the other hand, show that $z^{\Delta_+}$ gives a finite value, and hence a solution with these asymptotics is normalizable. With this action, show that the smallest conformal dimension $\Delta = \Delta_+$ of a scalar operator in the dual theory is $\Delta \geq \frac{D}{2}$.

b) Show that the following modified action

$$S_{KW}[\phi] \equiv -\frac{1}{2}\int d^{D+1}x\sqrt{g}\left(\phi(-\Box + m^2)\phi\right) = S_{\text{usual}}[\phi] - \int_{\partial AdS}\sqrt{\gamma}\phi\partial_n\phi.$$

produces the same bulk equations, but has the following property: in the window of mass values (we could call this the KW window)

$$-\frac{D^2}{4} < L^2m^2 < -\frac{D^2}{4} + 1,$$

*both* roots $\Delta_\pm$ give normalizable solutions with respect to the action $S_{KW}$ (*i.e.* $S_{KW}[z^{\Delta_\pm}] < \infty$).

What is the lowest conformal dimension you can obtain now? (Hint: you can now consider $\Delta_+$ to be the dimension of the source, and $\Delta_-$ to be the dimension of the operator.)

[3] This result is from Klebanov-Witten, hep-th/9905104.