MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2008

# Problem Set 5

*Finally, we can calculate*

**Due:** Tuesday, November 25, 2008.

1. **Bulk vector fields.**

   a) Use the inversion trick to show that

$$K_A^M(w,\vec{x}) = c_D \frac{w_0^{D-1}}{(w-\vec{x})^{2(D-1)}} J_A^M(w-\vec{x})$$

   (where $w^A \equiv (w_0, \vec{w})^A$ labels a point in the bulk of $AdS_{D+1}$) is the bulk-to-boundary propagator for a massless vector field, *i.e.* this object solves the bulk Maxwell equations, and $\to \delta^D(\vec{w}-\vec{x})$ as $w^0 \to 0$. Here

$$J_A^M(x) \equiv \delta_A^M - 2\frac{x_A x^M}{x^2}.$$

   Show that

$$J_A^M(x) = x'^2 \frac{\partial x_A}{\partial x_M'} \quad \text{with} \quad x^A = \frac{x'^A}{x^2}$$

   so this $J_A^M$ is the Jacobian for the inversion transformation.

   b) Check that the gravity calculation of three point function

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}^\star_\Delta(x_2)\mathcal{J}^\mu(x_3)\rangle$$

   ($\mathcal{J}^\mu$ is the conserved current to which the bulk gauge field $A_A$ couples) has the form required by conformal symmetry given in lecture.

   c) [extra bonus problem] Show that the form of the three point function above is determined by conformal invariance up to a constant.

2. **Relation between AdS propagators.**

   Show that the bulk-to-boundary and boundary-to-boundary propagators for a scalar field in $AdS$ are related by

$$K^\Delta(z,x;x') = \lim_{z'\to\epsilon} \frac{\Delta_+ - \Delta_-}{\epsilon^\Delta} G^\Delta(z,x;z',x').$$

Hint: use 'Green's second identity'

$$\int_U \sqrt{g}\left(\phi(\Box - m^2)\psi - ((\Box - m^2)\phi)\psi)\right) = \int_{\partial U} \sqrt{\gamma}\left(\phi n \cdot \partial\psi - (n \cdot \partial\phi)\psi\right) \quad \forall \phi, \psi$$

with $\phi = G, \psi = K$. The $G$ appearing in this relation is defined to be the *normalizable* solution to the wave-equation-with-source

$$\left(\Box_x - m^2\right) G(z, x; z', x') = \frac{1}{\sqrt{g}} \delta^D(x - x')\delta(z - z')$$

which is regular in the interior; $K$ is defined to be the solution to the homogeneous wave equation which is regular in the interior and approaches

$$\lim_{z \to \epsilon} K^\Delta(z, x; x') = \epsilon^{\Delta_-} \delta^D x - x'.$$

[Note: it is possible to show this using properties of the hypergeometric function appearing in the explicit expression for $G$; this proof is not so illuminating and requires actually knowing the exact solution]

3. **Wilson line with cusp.**

   Use the AdS/CFT duality to compute the strong-coupling behavior of

   $$\langle W[v] \rangle_{CFT},$$

   *i.e.* the vev of a Wilson loop (in the fundamental) associated to a curve described by a 'v' of opening angle $\theta$, in a CFT with a $AdS$ string dual. Show that the renormalized expecation value behaves like

   $$\ln \langle W[v] \rangle^{ren}_{CFT} \sim \ln \frac{L}{\epsilon} \sqrt{4\pi\lambda}\ \Gamma(\theta)$$

   where $\Gamma$ is some function of the opening angle which vanishes as $\theta$ approaches $\pi$ (this means that the discontinuity in the line is small). $\epsilon$ and $L$ are UV and IR cutoffs on the radial coordinate in AdS (*i.e.* $\epsilon < z < L$ in the coordinates we have been using).

4. **Surface gravity.**

   Compute the periodicity of $y$ for which this metric is regular at $z = z_m$ (*i.e.* has no conical deficit):

   $$ds^2 = \Omega(z)\left(f(z)dy^2 + \frac{dz^2}{f(z)} + ds^2_{\text{other}}\right)$$

where $f(z)$ is a function with a first-order zero at $z = z_m$ ($i.e.$ $\partial_z f(z_m) \neq 0$ ) and $\Omega(z)$ is regular and non-vanishing at $z = z_m$.

If we think of $y$ as imaginary time, this periodicity determines the temperature of the black hole, since finite temperature means periodic Euclidean time.

a) Specialize your answer to the case of the euclidean Schwarzchild black hole in flat space, for which

$$f(z) = 1 - \frac{2GM}{z},\ \Omega = 1,\ ds^2_{\text{other}} = z^2 d\vec{x}^2.$$

b) Specialize your answer to the case of the euclidean AdS black hole (with planar horizon), for which

$$f(z) \equiv 1 - \frac{z^4}{z_m^4},\ \Omega = \frac{1}{z^2},\ ds^2_{\text{other}} = d\vec{x}^2.$$

This geometry is also relevant to the model of confinement obtained by compactifying a supersymmetric gauge theory on a circle with supersymmetry-breaking (Scherk-Schwarz) boundary conditions.

c) Show that you get the same answer by computing the 'surface gravity' $\kappa$ of the horizon (the locus $z = z_m$), which can be defined by

$$\kappa^2 \equiv \frac{1}{2} \nabla_a \xi^b \nabla_c \xi^d g_{bd} g^{ac}|_{z=z_m}$$

where $\xi^a$ is the tangent vector to the shrinking circle, $\xi = \partial_y$.