8.821 F2008 Problem Set 3 Solutions

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I. THE SECRET IDENTITY OF THE CONFORMAL ALGEBRA

Recall the conformal algebra[1] for $\mathbb{R}^{p,q}$ is given by translations P_μ, rotations $L_{\mu\nu}$, special conformal transformations K_μ, and dilatations D, satisfying the algebra

$$[D, P_\mu] = iP_\mu \qquad [D, K_\mu] = -iK_\mu \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \tag{1}$$

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \quad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \tag{2}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \tag{3}$$

Of this algebra really only the first line should be unfamiliar, as the second line simply says that P_μ and K_μ are vectors and the last line is just the Lorentz algebra $SO(p,q)$. Now to put this into a more intuitive form we imagine that we are in a bigger space with two extra coordinates which we call -1 and 0 (the remaining coordinates are then $1...p+q$). Now consider the algebra defined by the following antisymmetric tensor generators J_{ab} (where a, b include the two new coordinates and μ, ν run only over the original $p+q$)

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{4}$$

$$J_{-1,0} = D \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{5}$$

I claim these generators secretly obey the algebra of $SO(p+1, q+1)$:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{6}$$

where η_{ab} is the metric tensor for $\mathbb{R}^{p+1,q+1}$. To verify this, we compute commutators; it is clear that those of $J_{\mu\nu}$ with itself will work out. The next easiest is $[J_{\mu\nu}, J_{-1,0}]$; if this is part of a Lorentz algebra it should vanish from (6) as these two generators have no indices in common, and indeed it does vanish (in the lower dimensional interpretation this is because D is a scalar on the original $\mathbb{R}^{p,q}$). The rest require slightly more work:

$$\begin{aligned}[J_{0\rho}, J_{\mu\nu}] &= i(\eta_{\rho\nu}J_{0\nu} - \eta_{\rho\nu}J_{0\nu}) \\ &= i(\eta_{\rho\nu}J_{0\nu} + \eta_{0\nu}J_{\rho\mu} - \eta_{0\mu}J_{\rho\nu} - \eta_{\rho\nu}J_{0\mu}),\end{aligned} \tag{7}$$

where I have introduced the objects $\eta_{0\nu}$ to make more clear the comparison with (6) ; this is indeed exactly what we expect. A precisely analogous computation can be done for $J_{-1,\mu}$. Slightly more interesting are the following

$$[J_{-1,0}, J_{-1,\mu}] = \left[D, \frac{1}{2}(P_\mu - K_\mu)\right] = \frac{i}{2}(P_\mu + K_\mu) = iJ_{0,\mu} = -i\eta_{-1,-1}J_{0\mu} \tag{8}$$

$$[J_{-1,0}, J_{0,\mu}] = \left[D, \frac{1}{2}(P_\mu + K_\mu)\right] = \frac{i}{2}(P_\mu - K_\mu) = iJ_{-1,\mu} = i\eta_{00}J_{0\mu} \tag{9}$$

$$\tag{10}$$

In the last equality I have put in the expected value from (6). Note that this only works out if $\eta_{-1,-1} = -1$ and $\eta_{00} = +1$; thus the enlarged space has one extra time and one extra spatial dimension, as claimed earlier. There remains only one more

$$[J_{-1,\mu}, J_{0\nu}] = \frac{1}{4}[P_\mu - K_\mu, P_\mu + K_\nu] = \frac{i}{2}[-2\eta_{\mu\nu}D + L_{\nu\mu} + L_{\mu\nu}] = -i\eta_{\mu\nu}J_{-1,0}, \tag{11}$$

again, as expected from (6). Thus the conformal group on $\mathbb{R}^{p,q}$ is isomorphic to $SO(p+1, q+1)$! Note that this fits nicely into our understanding of AdS/CFT; the conformal group from say a (Euclidean) CFT_d is realized as the isometries of AdS_{d+1}, a space that can be nicely embedded into $\mathbb{R}^{1,d+1}$ as a hyperboloid that is manifestly preserved under the action of $SO(1, d+1)$, which as we see is nothing but the conformal group of the CFT_d.

[1] found e.g. on p98 of [1]

II. THE QUADRATIC-NESS OF CONFORMAL TRANSFORMATIONS

Here we will prove that the generators ϵ_μ of an infinitesimal conformal transformation, which by definition satisfy

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial \cdot \epsilon \tag{12}$$

are at most quadratic in x, for $d > 2$. To start, we dot the formula with ∂^μ and get

$$\Box \epsilon_\nu + \left(1 - \frac{2}{d}\right)\partial_\nu(\partial \cdot \epsilon) = 0 \tag{13}$$

Note that for $d = 2$ we find only that $\Box\epsilon_\nu = 0$, so any harmonic function generates a conformal transformation. Now we take a derivative ∂_μ of the above equation and add to it the same equation with μ and ν interchanged, resulting in

$$\Box(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + 2\left(1 - \frac{2}{d}\right)\partial_\mu\partial_\nu(\partial \cdot \epsilon) = 0 \tag{14}$$

Now plugging in (12) we find that

$$\left[\frac{1}{d}\eta_{\mu\nu}\Box + \left(1 - \frac{2}{d}\right)\partial_\mu\partial_\nu\right]\partial \cdot \epsilon = 0 \tag{15}$$

Hitting this with $\eta^{\mu\nu}$ we find soothingly that $(2 - 2/d)\Box\partial \cdot \epsilon = 0$. Assuming that $d \neq 1$, we find that

$$\Box\partial \cdot \epsilon = 0 \tag{16}$$

Though this appears to be a scalar equation, it turns out that this is all we need, essentially because equations like (12) mean that the tensor structure of derivatives of ϵ is entirely determined by scalar functions like $\partial \cdot \epsilon$. More concretely, plug (16) into (15) to find (for $d \neq 2$) that in fact $\partial_\mu\partial_\nu(\partial \cdot \epsilon) = 0$. Finally now take two derivatives $\partial_\rho\partial_\sigma$ of (12); the right hand side vanishes by the relation that we just found, and the left hand side is

$$\partial_\rho\partial_\sigma(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = 0 \tag{17}$$

We are almost done; we just now need to show that not only the symmetric combination $(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)$ but also the antisymmetric combination $(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)$ has vanishing second derivatives. To that end, subtract from (17) itself with a relabeling of indices:

$$\partial_\rho\partial_\sigma(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) - \partial_\mu\partial_\sigma(\partial_\rho \epsilon_\nu + \partial_\nu \epsilon_\rho) = 0 \tag{18}$$

Some rearrangement and use of the commutativity of partials reveals that the first term in each bracketed expression cancels with its counterpart and we find

$$\partial_\sigma\partial_\nu(\partial_\rho \epsilon_\mu - \partial_\mu \epsilon_\rho) = 0 \tag{19}$$

Thus both the symmetric and antisymmetric combinations have vanishing second derivatives, and ϵ_μ is at most quadratic in x.

III. FUN WITH LARGE N

This problem was basically solved in class, but we'll review the argument here. Consider a very general matrix quantum field theory with the Lagrangian

$$\mathcal{L} = \int d^d x \mathrm{tr}(D_\mu\tilde{\Phi}D^\mu\tilde{\Phi} + g\tilde{\Phi}^3 + g^2\tilde{\Phi}^4 + ...) \tag{20}$$

where $\tilde{\Phi}$ is an $N \times N$ matrix and g is some sort of coupling. If we now redefine our fields: $\tilde{\Phi} = \Phi/g$, then all the factors of g actually come outside the Lagrangian:

$$\mathcal{L} = \frac{1}{g^2}\int d^d x \mathrm{tr}(D_\mu\Phi D^\mu\Phi + \Phi^3 + \Phi^4 + ...) \tag{21}$$

Now what is the contribution of a given Feynman diagram with V vertices, P propagators, and L loops? Note that with the field definitions used in (21), the propagator is the inverse of the operator D^2/g^2 and so contributes a factor of g^2; similarly each vertex (regardless of the type: cubic, quartic, etc.) contributes $1/g^2$. Finally each loop involves a trace and so contributes a factor of N. Thus the scaling of the diagram is

$$\text{contribution} \sim (g^2)^{(V-P)} N^L \sim N^{(L+V-P)} \lambda^{(V-P)} \tag{22}$$

where in the second equality I have written things in terms of the t'Hooft coupling $\lambda \equiv g^2 N$. Now however–note that we learned in lecture how to associate a triangulation of a two-dimensional surface Σ to this Feynman diagram; each propagator becomes an edge of the triangulation, each vertex is still a vertex, and each index loop becomes a face. Thus really we have

$$\text{contribution} \sim N^{\text{faces}-\text{edges}+\text{vertices}} \lambda^{\text{vertices}-\text{edges}} \tag{23}$$

Note that the combination faces - edges + vertices is called the *Euler characteristic* χ of the surface Σ; I say "of the surface" and not "of the triangulation" because of the remarkable fact that it is a topological invariant and depends only on the surface being triangulated, not the triangulation itself. Thus the scaling of the diagram with N depends only on the topology of the surface that the diagram represents. Note also that the dependence on the t'Hooft coupling λ on the other hand does depend on the triangulation (vertices - edges is not a topological invariant) and so actually depends on the connectedness of the Feynman diagram being considered.

Anyway, for a sphere $\chi = 2$; thus we see that for all diagrams with the topology of a sphere ("planar diagrams") we get a contribution that goes like N^2.

IV. USEFUL COORDINATES IN ADS

We define Lorentzian AdS_{p+2} to be the set of points X^a inside $\mathbb{R}^{p+1,2}$ satisfying

$$-L^2 = \eta_{ab} X^a X^b \equiv -(X^{p+2})^2 - (X^0)^2 + \sum_{i=1}^{p+1} (X^i)^2 \tag{24}$$

Here $\mathbb{R}^{p+1,2}$ is a flat space with *two time directions*, so its metric is given by $\eta_{ab} = \mathrm{diag}(-,-,+,+..+)$. For most of the remaining problems we will need to compute induced metrics; for reference, recall that if we define a subspace in terms of embedding coordinates x^i by specifying functions $X^a(x^i)$, then the induced metric g_{ij} on the surface parametrized by x^i

$$g_{ij} = G_{ab} \frac{\partial X^a}{\partial x^i} \frac{\partial X^b}{\partial x^j}, \tag{25}$$

where G_{ab} is the metric on the larger space (i.e. η_{ab} in our case).

A. Global Coordinates

We take first the coordinate choice

$$X^{p+2} = L\cosh(\rho)\sin(\tau) \tag{26}$$
$$X^0 = L\cosh(\rho)\cos(\tau) \tag{27}$$
$$X^i = L\sinh(\rho)\Omega_i \tag{28}$$

where Ω_i are functions on the p-sphere such that $\sum_i (\Omega_i)^2 = 1$. To compute the induced metric we first require various partial derivatives, which are very easily found to be

$$\frac{\partial X^{p+2}}{\partial \rho} = L\sinh(\rho)\sin(\tau) \qquad \frac{\partial X^{p+2}}{\partial \tau} = L\cosh(\rho)\cos(\tau) \tag{29}$$
$$\frac{\partial X^0}{\partial \rho} = L\sinh(\rho)\cos(\tau) \qquad \frac{\partial X^0}{\partial \tau} = -L\cosh(\rho)\sin(\tau) \tag{30}$$
$$\frac{\partial X^i}{\partial \rho} = L\cosh(\rho)\Omega_i \qquad \frac{\partial X^i}{\partial \theta_j} = L\sinh(\rho)\frac{\partial \Omega^i}{\partial \theta^j} \tag{31}$$
$$\tag{32}$$

Note for example that to find the component $g_{\rho\rho}$ we want from (25)

$$g_{\rho\rho} = \eta_{ab}\frac{\partial X^a}{\partial\rho}\frac{\partial X^b}{\partial\rho} = -\left(\frac{\partial X^{p+2}}{\partial\rho}\right)^2 - \left(\frac{\partial X^0}{\partial\rho}\right)^2 + \sum_{i=1}^{p+1}\left(\frac{\partial X^i}{\partial\rho}\right)^2 = L^2(\cosh^2(\rho) - \sinh^2(\rho)) = L^2 \tag{33}$$

Similarly, we find

$$g_{\tau\tau} = -L^2\cosh^2(\rho)(\cos^2(\tau) + \sin^2(\tau)) = -L^2\cosh^2(\rho) \tag{34}$$

and the mixed term $g_{\rho\tau}$ is easily seen to vanish. For the angles parametrizing the p-sphere we find

$$g_{ij} = L^2\sum_k \sinh^2(\rho)\frac{\partial\Omega_k}{\partial\theta_i}\frac{\partial\Omega_k}{\partial\theta_j} \tag{35}$$

However this is by definition simply the metric on a p-sphere with radius $L\sinh(\rho)$. Thus the metric of global AdS is

$$ds^2 = L^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\Omega_p^2) \tag{36}$$

B. Poincare patch

Here we define a set of coordinates $\{u, x^\mu\}$ via

$$X^{p+2} + X^{p+1} = u \tag{37}$$

$$-X^{p+2} + X^{p+1} = v \tag{38}$$

$$X^\mu = \frac{ux^\mu}{L} \tag{39}$$

Now the first step is to figure out what v is on the AdS surface; to do this we plug the above equations into the defining equation (24) and solve for v to get

$$v = -\frac{L^2}{u} - \frac{u}{L^2}\eta_{\mu\nu}x^\mu x^\nu \tag{40}$$

We can now explicitly write X^{p+2} and X^{p+1} in terms of the intrinsic coordinates and get

$$X^{p+1} = \frac{1}{2}\left(u - \frac{L^2}{u} - \frac{u}{L^2}\eta_{\mu\nu}x^\mu x^\nu\right) \qquad X^{p+2} = \frac{1}{2}\left(u + \frac{L^2}{u} + \frac{u}{L^2}\eta_{\mu\nu}x^\mu x^\nu\right) \tag{41}$$

The remainder of the problem simply involves taking partial derivatives of the X^a as before. This is not terribly interesting: I present some intermediate steps

$$\frac{\partial X^{(p+1),(p+2)}}{\partial x^\mu} = \mp\frac{u}{L^2}x_\mu \qquad \frac{\partial X^{(p+1),(p+2)}}{\partial u} = \frac{1}{2}\left(1 \pm \left[\frac{L^2}{u^2} - \frac{1}{L^2}\eta_{\mu\nu}x^\mu x^\nu\right]\right) \tag{42}$$

$$\frac{\partial X^\mu}{\partial u} = \frac{x^\mu}{L} \qquad \frac{\partial X^\mu}{\partial x^\nu} = \delta^\mu_\nu\frac{u}{L} \tag{43}$$

Two signs indicate that the top is for $p+1$ and the bottom for $p+2$. From now putting these into the formula (25) results after some honestly straightforward algebra in

$$ds^2 = \frac{u^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{u^2}du^2 \tag{44}$$

Finally, we should worry about what portion of the hyperboloid is covered by these coordinates; noting indeed that the norm of the Killing vector ∂_{x^0} vanishes at $u = 0$, we see that we cannot actually move to negative u; thus we are confined to the portion of the hyperboloid where $X^{p+2} + X^{p+1} > 0$, implying that we cover only half of the hyperboloid.

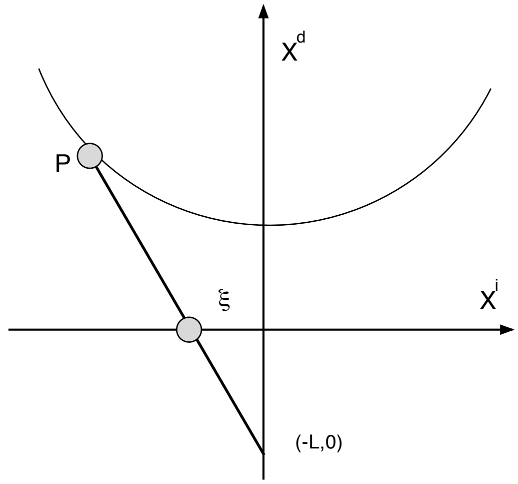

FIG. 1: Picture illustrating embedding for stereographic coordinates

C. Yet Another Coordinate System

Consider now the new variable $r \equiv L\sinh(\rho)$ where ρ is the global coordinate used in (57). We have $dr = L\cosh(\rho)d\rho$ and thus the metric (57) becomes

$$ds^2 = -L^2\cosh^2(\rho)d\tau^2 + \frac{dr^2}{\cosh^2(\rho)} + L^2\sinh^2(\rho)d\Omega_p^2 \tag{45}$$

Now using $\cosh^2(\rho) = 1 + \sinh^2(\rho) = 1 + r^2/L^2$ we obtain

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_p^2 \tag{46}$$

where I have introduced a dimensional time $t \equiv L\tau$. This is the claimed form. Now let us think about how to put a black hole in this background; to add a black hole without adding any extra matter, we would like to somehow deform the metric without altering the values of the Einstein tensor (which determines what the supporting stress energy must be). Note that if indeed $G_{\theta\theta} \sim \partial_r(r^p\partial_r H)$, then there are two types of terms we can add to H without changing the value of $G_{\theta\theta}$: a constant term and a term going like $1/r^{p-1}$. The constant term is not physically relevant (as it can essentially be absorbed into a coordinate redefinition), so we are led to the following form for the metric of Schwarzschild-AdS_{p+2}:

$$ds^2_{\mathrm{SchAdS}} = -\left(1 - \frac{2M}{r^{p-1}} + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r^{p-1}} + \frac{r^2}{L^2}} + r^2 d\Omega_p^2 \tag{47}$$

Note this becomes the familiar Schwarzschild metric if we set $p \to 2$ and smooth out the AdS-ness by putting $L \to \infty$. $2M$ is a parameter that is indeed related to the mass of the black hole. Keeping L finite, we see that far from the black hole this spacetime asymptotes to AdS_{p+2}. As we shall soon see in lecture, this metric is thought to represent the boundary CFT on $S^p \times \mathbb{R}$ in a thermal state with temperature given by the Hawking temperature of the black hole.

V. STEREOGRAPHIC PROJECTION COORDINATES

We take the Euclidean signature d-dimensional hyperbolic space H_d to be defined by the set of points

$$H_d = \left\{-X_d^2 + \sum_{i=0}^{d-1} X^i X^i = -L^2\right\} \subset \mathbb{R}^{d,1} \tag{48}$$

with metric on $\mathbb{R}^{d,1}$ given by $ds^2 = -dX_d^2 + \sum_{i=0}^{d-1} dX^i dX^i$ (note the labeling of coordinates here is slightly different from that in the problem set). There is a cute way to introduce coordinates on this space; consider a point P on the hyperboloid and and shoot a line from this to the point $(X^d, X^i) = (-L, \vec{0})$. This line will intersect the plane $X^d = 0$ at a point $(0, \xi^i)$; we define $r \equiv \sqrt{\xi^i \xi^i}$, but for simplicity we now rotate the coordinate system so that ξ^i points only in the $i = 0$ direction, so that $r = \xi^0$. Our whole problem is now in the (X^d, X^0) plane. Now we name this newly constructed line C and note that it is the set of points

$$C = \left\{ \begin{pmatrix} -L \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} L \\ r \end{pmatrix}, \lambda \in \mathbb{R} \right\} \subset \mathbb{R}^{d,1} \tag{49}$$

Where does this line hit the hyperboloid? The X^0 coordinate of this point can be seen from (48) and (49) to be the solution to the equation

$$\frac{X^0}{r} L - L = \sqrt{(X^0)^2 + L^2} \tag{50}$$

which upon restoration of the i index has the solution

$$X^i = \xi^i \frac{2L^2}{L^2 - r^2} \qquad r \equiv \sqrt{\xi^i \xi^i}, \tag{51}$$

precisely the coordinate choice made in the problem set. Figure 1 illustrates this construction.

The defining equation then implies that the remaining coordinate in the base space X^d is given by

$$X^d = L \frac{L^2 + r^2}{L^2 - r^2} \tag{52}$$

To compute the induced metric we now take various partial derivatives as before. Those of X^i are pretty immediate,

$$\frac{\partial X^i}{\partial \xi^j} = \frac{2L^2}{L^2 - r^2} \left(\delta^{ij} + \frac{2\xi^i \xi^j}{L^2 - r^2} \right), \tag{53}$$

whereas those of X^d require slightly more work, and are perhaps more easily found by taking a derivative of the defining equation (48) with respect to ξ^j

$$\begin{aligned} 2X_d \frac{\partial X_d}{\partial \xi^j} = 2 \sum_i X^i \frac{\partial X^i}{\partial \xi^j} \quad \rightarrow \quad \frac{\partial X_d}{\partial \xi^j} &= \frac{1}{X^d} \sum_i \xi^i \left(\frac{2L^2}{L^2 - r^2} \right)^2 \left(\delta^{ij} + \frac{2\xi^i \xi^j}{L^2 - r^2} \right) \\ &= \frac{\xi^j}{L} \left(\frac{2L^2}{L^2 - r^2} \right)^2 \end{aligned} \tag{54}$$

We now use (25) and plug in these results, obtaining

$$ds^2 = \left[-\frac{\xi^i \xi^j}{L^2} \left(\frac{2L^2}{L^2 - r^2} \right)^4 + \left(\frac{2L^2}{L^2 - r^2} \right)^2 \left(\delta^{li} + \frac{2\xi^l \xi^i}{L^2 - r^2} \right) \left(\delta^{mj} + \frac{2\xi^m \xi^j}{L^2 - r^2} \right) \right] d\xi^i d\xi^j \tag{55}$$

Expanding this out and waiting for the dust to settle, we obtain eventually

$$ds^2 = \left(\frac{2L^2}{L^2 - r^2} \right)^2 d\xi^i d\xi^i \tag{56}$$

as claimed.

VI. GEODESICS IN ADS

For this problem we will use the metric in global coordinates given in (57):

$$ds^2 = L^2 (-\cosh^2(\rho) dt^2 + d\rho^2 + \sinh^2(\rho) d\Omega_p^2) \tag{57}$$

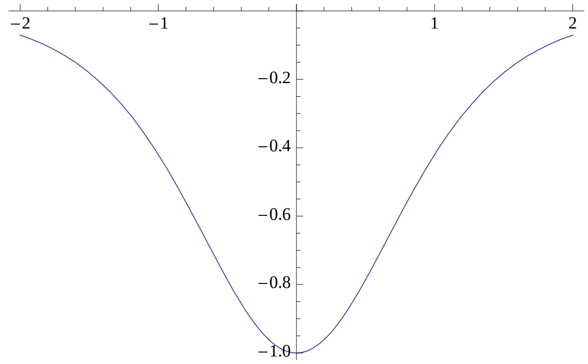

FIG. 2: Effective potential $V(\rho)$ for massive geodesic in AdS

We would first like to find the trajectory of a massless geodesic, starting from a point $\rho = \rho_0$. Along such a trajectory, $ds^2 = 0$, implying that

$$\frac{d\tau}{d\rho} = \frac{1}{\cosh(\rho)} \quad \to \quad \tau(\rho) = 2\arctan\left[\tanh\left(\frac{\rho}{2}\right)\right] - 2\arctan\left[\tanh\left(\frac{\rho_0}{2}\right)\right], \tag{58}$$

where I have adjusted the integration constant so that $\tau(\rho_0) = 0$. Now the coordinate time for a geodesic to leave ρ_0, go to the boundary, and come back is simply $2\tau(\infty)$, which means that the *proper* time elapsed on a stationary observer's clock is

$$\text{proper time} = 2L\cosh(\rho_0)\tau(\infty) = 2L\cosh(\rho_0)\left(\pi - 2\arctan\left[\tanh\left(\frac{\rho_0}{2}\right)\right]\right) \tag{59}$$

Note that this is finite regardless of the value of ρ_0; no matter where you are, a geodesic takes a finite amount of time to leave you and come back. This is a signature of the badness of the Cauchy problem in AdS, as an observer can communicate with regions that are infinitely far away in finite time.

Next, we would like to understand the behavior of a massive geodesic. To compute this without any mucking around with Christoffel symbols, note that for a massive particle the norm of the $p+2$-velocity $u^\mu \equiv dx^\mu/dT$ is always -1, where T is the proper time along the worldline. This means that

$$-\cosh^2(\rho)\left(\frac{d\tau}{dT}\right)^2 + \left(\frac{d\rho}{dT}\right)^2 = -\frac{1}{L^2} \tag{60}$$

On the other hand, we also know from old-fashioned GR [2] that the scalar obtained by dotting a Killing vector into a $(p+2)$-velocity is conserved along a geodesic. For us a Killing vector reflecting time translational invariance is given by $\xi = \partial_\tau$, (i.e. $\xi^\mu = \delta^\mu_\tau$), implying that $u_\tau = $ const along the geodesic. Calling the constant $-A$, we obtain

$$\cosh^2(\rho)\frac{d\tau}{dT} = A \tag{61}$$

The combination of (60) and (61) results in

$$\frac{1}{2}\left(\frac{d\rho}{dT}\right)^2 + \left(\frac{-A^2}{2\cosh^2(\rho)}\right) = -\frac{1}{2L^2} \tag{62}$$

This is nothing but the equation for conservation of energy for a particle of unit mass moving in a one-dimensional potential $V(\rho) = \left(\frac{-A^2}{2\cosh^2(\rho)}\right)$ with negative "energy" $-\frac{1}{2L^2}$. The potential is graphed above in (2); notice that since the particle always has negative "energy" it is bound to the well and can never reach infinity, regardless of the value of A.

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[2] see e.g. p136 of [2]

VII. VOLUMES AND AREAS ARE NOT SO DIFFERENT IN ADS...

...by which we mean that they scale the same way in the large ρ limit. To see this, consider some fixed $\bar{\rho}$ and compute the area $A(\bar{\rho})$ and volume $V(\bar{\rho})$ inside it. The area is immediately read off of (57) to be

$$A(\bar{\rho}) = \mathrm{Vol}(S^p) L^p \sinh^p(\bar{\rho}) \tag{63}$$

To compute the volume I assume for simplicity that $\bar{\rho}$ is big enough that I can take $\sinh(\rho)$ to be just $\exp(\rho)/2$. The volume is then just

$$V(\bar{\rho}) = \int_0^{\bar{\rho}} d\rho L A(\rho) \sim \frac{L^{p+1}}{2p} \mathrm{Vol}(S^p) \exp(p\bar{\rho}) \tag{64}$$

Thus $A(\bar{\rho})/V(\bar{\rho}) \to p/L$, as $\bar{\rho} \to \infty$, which is indeed finite.

[1] P. D. Francesco, P. Mathieu, and D. Senechal, "Conformal Field Theory," Springer, 1997.
[2] S. Carroll, "Spacetime and Geometry: An Introduction to General Relativity," Addison-Wesley, San Francisco 2004.