8.821 F2008 Problem Set 5 Solutions

Nabil Iqbal

I.   BULK VECTOR FIELDS

In this problem we will evaluate a certain three-point function using AdS/CFT, to verify that it has the form required by conformal symmetry. This will require some machinery, and I will follow the approach taken in [1]. We work with AdS_{d+1} with metric given by

$$ds^2 = \frac{1}{x_0^2}\left(dx_0^2 + \sum_{i=1}^{d} dx^i dx^i\right) \equiv \frac{1}{x_0^2} dx_A dx^A \tag{1}$$

For this problem we will work in Euclidean signature i runs over all d dimensions of the boundary theory, and $x^A = x_A = (x_0, x^i)$.

A.   Warmup

First, we will study the gauge field bulk-boundary propagator K_{MN}. The equations of motion for a massless gauge field in AdS_{d+1} were worked out in Problem Set 4. If we assume there is no momentum in any of the field theory directions, these equations reduce to (see (40) in Problem Set 4 solutions)

$$x_0^2 \partial_0 A_i + (3-d)x_0 \partial_0 A_i = 0 \tag{2}$$

We then have the immediate solution

$$A_i = \epsilon_i x_0^{d-2} \tag{3}$$

where ϵ_i is some polarization vector (we work in a gauge where $A_z = 0$). Now let us consider the effect of the infamous inversion on this solution. Recall that the inversion is a relation between coordinate systems

$$x^A = \frac{\bar{x}^A}{\bar{x}^A \bar{x}_A} \tag{4}$$

I denote the coordinates of the gauge field in the new barred coordinate system by $\bar{A}_B$; these take the form

$$\bar{A}_B(\bar{x}) = A_C(x(\bar{x}))\frac{\partial x^C}{\partial \bar{x}^B} \tag{5}$$

Working out the Jacobian of the coordinate transformation (done previously in Problem Set 4, problem 1) we find that the solution (3) becomes

$$\bar{A}_B(\bar{x}^C) = \frac{\bar{x}_0^{d-2}}{\bar{x}^{2(d-1)}}\left(\delta_B^A - \frac{\bar{x}^A \bar{x}_B}{\bar{x}^2}\right)\delta_A^i \epsilon_i \tag{6}$$

This is precisely the bulk-boundary propagator from a point in the bulk x^C to the point $x^i = 0$ on the boundary. This can be verified in exactly the same way as was done in class for the scalar case. To recap briefly: by construction this is a solution to the equations of motion. Furthermore at $x_0 = 0$ the propagator vanishes at all points where $x^i \neq 0$, and a scaling argument shows that $\int d^d x A_i(x)$ is independent of x_0 and proportional to ϵ_i, showing that the boundary value of this propagator is a delta function, as desired. Thus, dropping the bars, the bulk-to-boundary propagator from an arbitrary point y^i at the boundary to a point x^A in the bulk is just

$$K_{Ai}(x^A; y_i) = c\frac{x_0^{d-2}}{(x_0^2 + (x-y)^i(x-y)_i)^{d-1}}\left(\delta_{Ai} - \frac{(x-y)_A(x-y)_i}{x^2}\right) \tag{7}$$

This is (almost) the form claimed in the problem set. As it turns out, we will not actually require this explicit form for the rest of the computation.

B.   3-point function

Next, we turn to the computation of the following CFT three-point function

$$\langle \mathcal{O}(x_1)\mathcal{O}^*(x_2)J_i(x_3)\rangle_{\text{CFT}} \tag{8}$$

Here J_i is a symmetry current and $\mathcal{O}$ is an operator of dimension Δ that is charged under that symmetry. Let us take J_i to be a $U(1)$ current for simplicity; it is then dual to a bulk $U(1)$ gauge field A_M, and $\mathcal{O}$ is dual to a bulk massive scalar field ϕ that is charged under that $U(1)$. The covariant derivative of this field ϕ is then

$$D_M\phi = (\partial_M - igA_M)\phi \tag{9}$$

Thus the bulk scalar action contains a term

$$S_{\text{bulk}} \supset -igA_M(\phi\partial_M\phi^* - \phi^*\partial_M\phi) \tag{10}$$

This is the term that will be relevant for us. Indeed, as discussed in lecture it is rather clear that the three point function that we want is proportional to

$$\langle \mathcal{O}(x_1)\mathcal{O}^*(x_2)J_i(x_3)\rangle_{\text{CFT}} \sim g\int d^{d+1}z\sqrt{-g}g^{MN}K_{Mi}(z;x_3)\left[K_\Delta(z;x_1)\partial_N K_\Delta(z;x_2) - K_\Delta(z;x_2)\partial_N K_\Delta(z;x_1)\right], \tag{11}$$

where K_{Mi} is the gauge field bulk-boundary propagator found above and $K_\Delta(z;x_i)$ is the familiar scalar dimension-Δ bulk-boundary propagator

$$K_\Delta(z,x^i) = c_\Delta\left(\frac{z_0}{z_0^2 + (z-x)^i(z-x)_i}\right)^\Delta \tag{12}$$

At first glance, this integral looks completely and utterly impossible. We are going to do it anyway. We make our lives simpler with the following steps:

1. First, set $x_3 = 0$. It should be clear there is no loss of generality here.

2. Next, we stare hard at the right-hand side of (11). Clearly we can do this integral in any coordinate system we want; let us thus do it in a coordinate system *related to the one where $x_3 = 0$ by an inversion*. Why does this help us? After the inversion, the point x_3 (which is at the AdS boundary) becomes the point at infinity; but this means that for the gauge field propagator we can use the hilariously simple solution (3) rather than the messy (7). Let us thus work out how everything on the right-hand side of (11) transforms under an inversion $x^A = \frac{\bar{x}^A}{\bar{x}^A\bar{x}_A}$:

$$(x-y)^2 = \left(\frac{\bar{x}}{\bar{x}^2} - \frac{\bar{y}}{\bar{y}^2}\right)^2 = \left(\frac{1}{\bar{x}^2} + \frac{1}{\bar{y}^2} + \frac{(\bar{x}-\bar{y})^2 - (\bar{x}^2+\bar{y}^2)}{\bar{x}^2\bar{y}^2}\right) = \frac{(\bar{x}-\bar{y})^2}{\bar{x}^2\bar{y}^2} \tag{13}$$

$$K_\Delta(z;x^i) = \left(\frac{\bar{z}_0\bar{z}^2\bar{x}^2}{\bar{z}^2(\bar{z}-\bar{x})^2}\right)^\Delta = K_\Delta(\bar{z};\bar{x}^i)\,\bar{x}^{2\Delta} \tag{14}$$

Note that in the first formula x and y are vectors x^A, y^A labeling a point in AdS and in the second $x = (0, x^i)$ should be viewed as a (d+1) dimensional vector but restricted to the boundary of AdS. Putting these pieces together the integral becomes

$$g(\bar{x}_1\bar{x}_2)^{2\Delta}\int d^{d+1}\bar{z}\frac{1}{\bar{z}}\left[\left(\frac{\bar{z}_0}{(\bar{z}-\bar{x}_1)^2}\right)^\Delta \partial_{z_i}\left(\frac{\bar{z}_0}{(\bar{z}-\bar{x}_2)^2}\right)^\Delta\right] - (x_1 \leftrightarrow x_2) \tag{15}$$

This looks more tractable.

3. Finally, we "do" an integral:

$$F(\vec{x},\vec{y}) \equiv \int_0^\infty dz_0\int d^d\vec{z}\frac{z_0^a}{(z_0^2 + (\vec{z}-\vec{x})^2)^b(z_0^2 + (\vec{z}-\vec{y})^2)^c} = \text{const}\times|\vec{x}-\vec{y}|^{1+a+d-2b-2c} \tag{16}$$

Note that it is clear from translational and scale invariance in $\vec{x}$, $\vec{y}$ that the answer can only be a power law in $|\vec{x}-\vec{y}|$, and dimensional analysis then tells us the exponent. To get the overall factor one must work much harder; I believe it can be done with Feynman parameters (eq (22) in [1]), but I am not going to do it here.

With this result our work is actually done; if we trade the derivative ∂_{z_i} for a derivative $\partial_{x_2^i}$ in (15) and use the integral result above, the answer is simply

$$\langle \mathcal{O}(x_1)\mathcal{O}^*(x_2)J_i(0)\rangle = \text{const} \times (\bar{x}_1\bar{x}_2)^{2\Delta}\partial_{\bar{x}_2^i}|\bar{x}_1 - \bar{x}_2|^{d-2\Delta} - (x_1 \leftrightarrow x_2) \tag{17}$$

Here the barred coordinates should be viewed as functions of the unbarred ones. Thus taking the derivative and expressing everything in terms of the unbarred coordinates, we obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}^*(x_2)J_i(0)\rangle = \text{const} \times \left(\frac{x_1}{x_1^2} - \frac{x_2}{x_2^2}\right)_i \frac{|x_1 - x_2|^{d-2\Delta-2}}{(|x_1||x_2|)^{d-2}} \tag{18}$$

It is clear how to extend this to points $x_3 \neq 0$; translational invariance means that we simply replace x_2 with $x_2 - x_3$, etc. This expression is exactly the form required by conformal invariance (as discussed in lecture), and it is a rather nontrivial check of the AdS/CFT correspondence that we could obtain this form from this somewhat involved gravity calculation.

II. RELATION BETWEEN ADS PROPAGATORS

Here we will establish a useful relation between bulk-boundary and bulk-to-bulk propagators in AdS space. Switching notation for simplicity, we will work now with AdS space in the form

$$ds^2 = \frac{1}{z^2}\left(dz^2 + (dx^i)^2\right) \tag{19}$$

Let $K_\Delta(z, x; x')$ be the bulk-boundary propagator defined in (12). This is a solution to the bulk equations of motion that satisfies

$$\lim_{z\to\epsilon} K_\Delta(z, x; x') = \epsilon^{\Delta_-}\delta^d(x - x') \tag{20}$$

and let $G_\Delta(z, x; z', x')$ be the bulk-bulk propagator, defined as the normalizable solution to the bulk equations of motion, but with source:

$$(\Box_x - m^2)G(z, x; z', x') = \frac{1}{\sqrt{g}}\delta^d(x - x')\delta(z - z') \tag{21}$$

Finally, we consider "Green's second identity", applicable to any two functions ϕ and ψ:

$$\int_U d^{d+1}x\sqrt{g}\left[\phi(\Box_x - m^2)\psi - \psi(\Box_x - m^2)\phi\right] = \int_{\partial U} d^d x\sqrt{\gamma}(\phi n \cdot \partial\psi - \psi n \cdot \partial\phi) \tag{22}$$

Now: set $\phi(z, x) = G_\Delta(z, x; z_1, x_1)$ and $\psi(z, x) = K_\Delta(z, x; x_2)$ in that case we end up with the following:

$$-K_\Delta(z_1, x_1; x_2) = \lim_{z\to\epsilon}\int_{\partial AdS} d^d x z^{1-d}\left[G_\Delta(z, x; z_1, x_1)\partial_z K_\Delta(z, x; x_2) - K_\Delta(z, x; x_2)\partial_z G_\Delta(z, x; z_1, x_1)\right] \tag{23}$$

Now, if z and z_1 are separated, then the fact that G is a normalizable solution to the AdS wave equation means that asymptotically it must behave as $G \sim z^{\Delta_+}$. Using this fact and the known behavior of the scalar propagator (20) to evaluate the radial derivatives, we end up with

$$-K_\Delta(z_1, x_1; x_2) = \epsilon^{\Delta_- - d}\int d^d x\left(G_\Delta(\epsilon, x; z_1, x_1)[\Delta_- - \Delta_+]\delta^d(x - x_2)\right) \tag{24}$$

Thus using $\Delta_+ + \Delta_- = d$, we finally end up with

$$K_\Delta(z_1, x_1; x_2) = \frac{\Delta_+ - \Delta_-}{\epsilon^{\Delta_+}}G_\Delta(\epsilon, x_2; z_1, x_1), \tag{25}$$

which is the desired relation.

III. WILSON LINE WITH CUSP

We are now going to compute the value of a Wilson loop with a "cusp" with opening angle v. This (and several related computations) have been done in [2], which I will mostly follow. We begin (as always) by reducing an impossible strongly coupled QFT problem to a fun classical mechanics exercise via the prescription discussed in lecture,

$$\langle W[v] \rangle_{\text{CFT}} = \exp(-S[v]) \tag{26}$$

where $S[v]$ is the classical action of a string worldsheet whose boundary on ∂AdS has the cusp with opening angle v. Note that this action is the normal Nambu-Goto action,

$$S[v] = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{27}$$

For this problem we will work with the AdS metric in the following form:

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + dr^2 + r^2 d\phi^2 + r^2 (\text{angles that will not matter to us}) \right) \tag{28}$$

We will use r, ϕ as coordinates on the string worldsheet as well; thus our task is to determine the function $z(r, \phi)$, which completely determines the embedding of the string in the bulk geometry. Now at first this problem looks very difficult: finding the string solution that asymptotes to a shape as peculiar as a line with a cusp seems like a nontrivial endeavor. Life is made easier by the fact that the boundary conditions are invariant under a scale transformation $r \to \lambda r$. This means that the bulk worldsheet solution should also be invariant under the bulk diffeomorphism $r \to \lambda r, z \to \lambda z$, leading us to the following scale-invariant ansatz for the string solution:

$$z(r, \phi) = \frac{r}{f(\phi)} \tag{29}$$

In that case the string action (27) becomes

$$S = \frac{L^2}{2\pi\alpha'} \int \frac{dr d\phi}{r} \sqrt{f^4 + f^2 + f'^2} \tag{30}$$

and we need only solve for $f(\phi)$. This action does not explicitly depend on ϕ; thus it admits a conserved "energy" E, which works out to be

$$E = \frac{f^2 + f^4}{\sqrt{f^4 + f^2 + f'^2}} \tag{31}$$

Finally, the boundary conditions for the cusp are

$$z(r, 0) = z(r, v) = 0 \tag{32}$$

This simply says that at the AdS boundary $z = 0$ the opening angle is v. Thus we have a setup where $f(\phi = 0)$ is ∞, it dips down to finite values for $\phi \in (0, v)$ before shooting off to ∞ again at $\phi = v$. We would like to relate the opening angle v to the parameter E which will be more useful for evaluating the action integral. To do this, note that the minimum value f_0 of $f(\phi)$ will happen at $\phi = v/2$, where $f'(v/2) = 0$ and thus we have

$$E = f_0 \sqrt{1 + f_0^2} \tag{33}$$

so we can write

$$\frac{v}{2} = \int_0^{v/2} d\phi = - \int_{f_0}^\infty df \left[f \sqrt{1 + f^2} \left(\frac{f^2(1 + f^2)}{f_0^2(1 + f_0^2)} - 1 \right)^{1/2} \right]^{-1} \tag{34}$$

where in the second integral we have changed integration variables from ϕ to f and used (31) and (33) to relate $df/d\phi$ to f_0. In principle this integral can be done as a function of f_0 and inverted to give f_0 as a function of v. In reality, this turns out to involve elliptic integrals and is rather unenlightening; thus let us instead expand in powers of f_0.

It is not immediately clear why you might want to do this, but if we press on we find that the leading term in the integral is

$$\frac{v}{2} = -\int_{f_0}^{\infty} df \, \frac{f_0}{f\sqrt{1+f^2}\sqrt{f^2+f^4}} + \mathcal{O}(f_0^2) = \arctan\left(\frac{1}{f_0}\right) + \mathcal{O}(f_0^2) \tag{35}$$

Thus we see that the limit $f_0 \to 0$ corresponds to $v \to \pi$, i.e. a small opening angle. We will work in this limit for the remainder of the problem (note that there is no problem in principle in extending this to arbitrary opening angle; however the number of elliptic integrals proliferates and so I prefer to avoid this).

Next we express the area in terms of f_0; starting with (30) and expanding in powers of f_0, we obtain eventually the very simple expression

$$S = \frac{L^2}{\pi\alpha'} \int \frac{dr}{r} \int_{f_0}^{\infty} df + \mathcal{O}(f_0^2) \tag{36}$$

This is divergent at large f; however if we look back at the definition of f in (29), we see that if we have a UV cutoff on z at $z = \epsilon$, we should really only integrate up to r/ϵ. Thus we find

$$S = \frac{L^2}{\pi\alpha'} \int dr \left(\frac{1}{\epsilon} - \frac{f_0}{r}\right) = \frac{\sqrt{\lambda}}{\pi}\left(\frac{1}{\epsilon}r_{\text{max}} - f_0 \log\left(\frac{r_{\text{max}}}{r_{\text{min}}}\right)\right) \tag{37}$$

We are done. The first term is proportional to the length of the Wilson line and is just the usual linear divergence; the second term is the extra contribution from the discontinuity in the Wilson line. Note it has the expected dependence on the IR and UV cutoffs r_{max} and r_{min} and indeed vanishes as $f_0 \to 0$, i.e. as $v \to \pi$ from (35).

IV.   SURFACE GRAVITY

Here we study the properties of metrics of the form

$$ds^2 = \Omega(z)\left(f(z)dy^2 + \frac{dz^2}{f(z)} + ds_{\text{other}}^2\right) \tag{38}$$

where the function $f(z)$ has a first-order zero at $z = z_m$ and $\Omega(z)$ is well-behaved there. We begin by Taylor expanding $f(z)$ near $z = z_m$,

$$ds^2 = \Omega(z_m)\left((z-z_m)f'(z_m)dy^2 + \frac{dz^2}{(z-z_m)f'(z_m)} + \ldots\right) \tag{39}$$

This peculiar-looking metric is actually something that we are very familiar with. To understand this, imagine picking a coordinate r such that

$$dz^2 \frac{\Omega(z_m)}{(z-z_m)f'(z_m)} = dr^2 \to (z-z_m) = \frac{f'(z_m)}{4\Omega(z_m)}r^2 \tag{40}$$

to obtain the second inequality I integrated the first relation in a small neighborhood of z_m. In terms of this new coordinate r, the metric near z_m simply becomes

$$ds^2 = \frac{r^2 f'(z_m)^2}{4}dy^2 + dr^2 + \ldots \tag{41}$$

This should look even more familiar if we define a coordinate $\theta = yf'(z_m)/2$, giving us simply

$$ds^2 = r^2 d\theta^2 + dr^2 + \ldots \tag{42}$$

This is precisely the metric of Euclidean 2d space in polar coordinates! However we know well that this spacetime has a conical singularity at the origin *unless* θ has a periodicity of 2π. This means that

$$y \in \left[0, \frac{4\pi}{f'(z_m)}\right] \tag{43}$$

Thus we have found the periodicity of y. Note that metrics such as (38) can arise as the analytic continuations of black holes to Euclidean space, in which case y is imaginary time and its periodicity is nothing less than the inverse temperature of the black hole.

We now work out some examples. First we do the analytic continuation of the ordinary 4d Schwarzschild black hole in flat space, for which we have

$$f(z) = 1 - \frac{2GM}{z} \qquad \Omega = 1 \tag{44}$$

Taking the derivative and evaluating at $z = z_m = 2GM$, we get $y_{\max} = 8\pi GM$ as the periodicity of y. Note this means the black hole has a temperature of $T = \frac{1}{8\pi GM}$, a rather famous result that we have quickly re-derived.

Next, we move on to the Euclidean AdS black hole with planar horizon, which has

$$f(z) = 1 - \left(\frac{z}{z_m}\right)^4 \qquad \Omega = \frac{1}{z^2} \tag{45}$$

From here we obtain $y_{\max} = \pi z_m$, and thus $T = \frac{1}{\pi z_m}$.

Finally we turn to a slightly different way to compute this periodicity using the *surface gravity* κ, defined by

$$\kappa^2 = \frac{1}{2}\nabla_a\xi^b\nabla_c\xi^d g_{bd}g^{ac}\Big|_{z=z_m} \tag{46}$$

where xi is the tangent vector to the shrinking circle, $\xi = \partial_y$. Note that the analysis of the earlier section showed us that the near-horizon metric could be written in the form

$$ds^2 = \alpha^2 r^2 dy^2 + dr^2 \tag{47}$$

where $\alpha = f'(z_m)/2$. Now to compute the relevant covariant derivatives we need the Christoffel symbols; the relevant components work out to be

$$\Gamma^y_{ry} = \frac{1}{r} \qquad \Gamma^r_{yy} = -r\alpha^2 \tag{48}$$

As the "upper" coordinates of ξ^a are all constant ($\xi^a = \delta^a_y$), only the Christoffel symbols contribute to the expression (46), and we obtain

$$\kappa^2 = \frac{1}{2}\left[(\Gamma^y_{yr}g_{yy}g^{rr})^2 + (\Gamma^r_{yy}g_{rr}g^{yy})^2\right] = \alpha^2 \tag{49}$$

Thus we find the following simple result for the surface gravity

$$\kappa = \frac{f'(z_m)}{2} = \frac{2\pi}{y_{\max}}, \tag{50}$$

telling us that the surface gravity is simply related to the inverse periodicity (and thus the temperature) of the black hole.

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[1] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, "Correlation Functions in the CFT_d/AdS_{d+1} correspondence", Nuclear Physics B, Volume 546, Number 1, 26 April 1999 , pp. 96-118(23) [arXiv:hep-th/9804058].
[2] N. Drukker, D. J. Gross, H. Ooguri, "Wilson Loops and Minimal Surfaces," Phys. Rev. D 60, 125006 (1999) [arXiv:hep-th/9904191]