University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Field Theory C (215C) Spring 2013
Assignment 1

Posted April 3, 2013 **Due 11am, Thursday, April 11, 2013**

Please remember to put your name at the top of your homework.

Announcements

- The 215C web site is:

  http://physics.ucsd.edu/∼mcgreevy/s13.

Problem Set 1

1. **Scale invariant quantum mechanics**

   Consider the action for one quantum variable $r \in \mathbb{R}^+$

   $$S[r] = \int dt \left(\frac{1}{2} m\dot{r}^2 - V(r) \right), \quad V(r) = \frac{\lambda}{r^2} .$$

   [Note: relative to earlier versions of this problem set, I've restored the (non-relativistic) mass parameter m. It can be eliminated by a multiplicative redefinition of the field r or of the time t. You should convince yourself that the physics of interest here should only depend on $m\lambda$.]

   (a) Show that the coupling λ is dimensionless: $[\lambda] = 0$.

   (b) Show that this action is *scale invariant*, *i.e.* show that under the transformation

   $$r(t) \to s^{\alpha} \cdot r(st) \tag{1}$$

   (for some α which you must determine), $s \in \mathbb{R}^+$ is a symmetry. Find the associated Noether charge $\mathcal{D}$. For this last step, it will be useful to note that the infinitesimal version of (1) is ($s = e^a, a \ll 1$)

   $$\delta r(t) = a \left(\alpha + t\frac{d}{dt} \right) r(t).$$

(c) Find the position-space Hamiltonian $\mathbf{H}$ governing the dynamics of r. Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2}\right)\psi_E(r) = E\psi_E(r).$$

Check that $[\mathbf{H}, \mathcal{D}] = 0$.

(d) Describe the behavior of the solutions to this equation as $r \to 0$. [Hint: in this limit you can ignore the RHS. Make a power-law ansatz: $\psi(r) \sim r^\Delta$ and find Δ.]

(e) What happens if $2m\lambda < -\frac{1}{4}$? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.

(f) A hermitian operator has orthogonal eigenvectors. We will show next that to make $\mathbf{H}$ hermitian when $2m\lambda < -\frac{1}{4}$, we must impose a constraint on the wavefunctions:

$$\left(\psi_E^\star \partial_r \psi_E - \psi_E \partial_r \psi_E^\star\right)|_{r=0} = 0 \tag{2}$$

There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point $r = 0$.

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by $\psi_{E'}^\star$ and integrate; multiply the second by $\psi_E^\star$ and integrate; take the difference. Show that the result is a boundary term which must vanish when $E = E'$.

(g) Show that the condition (2) is empty for $2m\lambda > -\frac{1}{4}$. Impose the condition (2) on the eigenfunctions for $2m\lambda < -\frac{1}{4}$. Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some silly reason, restricting the Hilbert space in this way is called a *self-adjoint extension*.] [1]

(h) [Extra credit] Consider instead a particle moving in $\mathbb{R}^d$ with a central $1/r^2$ potential, $r^2 \equiv \vec{x} \cdot \vec{x}$,

$$S[\vec{x}] = \int dt \left(\frac{1}{2}m\dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2}\right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in $\mathbb{R}^d$:

$$\vec{p}^2 = -\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r\right) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2}\hat{L}_{ij}\hat{L}_{ij}, \quad L_{ij} = -\mathbf{i}\left(x_i\partial_j - x_j\partial_i\right),$$

where $r^2 \equiv x^i x^i$. By 's-wave states' I mean those annihilated by $\hat{L}^2$.]

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[1] This model has been studied extensively, beginning, I think, with K.M. Case, *Phys Rev* **80** (1950) 797. More recent literature includes Hammer and Swingle, arXiv:quant-ph/0503074, Annals Phys. 321 (2006) 306-317. It also arises as the scalar wave equation for a field in anti de Sitter space.

2. **Gaussian integrals are your friend**

(a) Show that

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

(b) Consider a collection of variables $x_i, i = 1..N$ and a hermitian matrix a_{ij}. Show that

$$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a^{-1}_{ij} J^j}.$$

(Summation convention in effect, as always.)

[Hint: change into variables to diagonalize a. $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) Consider a Gaussian field X, governed by the (quadratic) action

$$S[x] = \int dt \frac{1}{2}\left(\dot{X}^2 - \Omega^2 X^2\right).$$

Show that

$$\langle e^{-\int ds J(s) X(s)} \rangle_X = \mathcal{N} e^{+\frac{1}{4}\int ds dt J(s) G(s,t) J(t)}$$

where G is the (Feynman) Green's function for X, satisfying:

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t).$$

Here $\mathcal{N}$ is a normalization factor which is independent of J. Note the similarity with the previous problem, under the replacement

$$a = -\partial_s^2 + \Omega^2, \quad a^{-1} = G.$$