Quantum Field Theory C (215C) Spring 2013
Assignment 2

Posted April 9, 2013 **Due 11am, Tuesday, April 23, 2013**

Relevant reading: Zee, Chapter I

Problem Set 2

1. **Spacelike particles mediate forces**

   Here you will get some practice with generating functionals, which we are going to be playing with a lot; this problem also contains some crucial physics.

   (a) [Banks, problem 2.11] Calculate the vacuum expectation value of the time-ordered exponential

   $$Z[J] \equiv \langle \mathcal{T} e^{\mathbf{i} \int J\phi} \rangle = \langle 0 | \mathcal{T} e^{\mathbf{i} \int d^D x J(x)\phi(x)} |0\rangle$$

   for a free relativistic scalar field ϕ, with action

   $$S[\phi] = \int d^D x \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right).$$

   Do this by functional methods, by performing the (gaussian) functional integral over ϕ, using the appropriate generalization of problem set 1 problem 2. You should find:

   $$Z[J] = e^{\frac{\mathbf{i}}{2} \int d^D x d^D y J(x) J(y) \int d^D p \frac{e^{-\mathbf{i} p(x-y)}}{p^2 - m^2 + \mathbf{i}\epsilon}} \, ,$$

   (b) [Banks, problem 2.11] Redo the previous calculation using canonical methods. By this I mean: write $\phi(x)$ as a sum of an operator involving only creation operators $\mathbf{a}_{\vec{p}}^\dagger$, and another with only annihilation operators $\mathbf{a}_{\vec{p}}$:

   $$\phi(x) = \int \frac{d^{D-1}\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \left(\mathbf{a}_{\vec{p}} e^{-\mathbf{i} px} + \mathbf{a}_{\vec{p}}^\dagger e^{\mathbf{i} px} \right)$$

   with $p^\mu \equiv (\omega_p, \vec{p})^\mu$ and $\omega_p \equiv \sqrt{\vec{p}^2 + m^2}$. Use the fact that annihilation operators annihilate $|0\rangle$. Compute enough terms in the power-series expansion in J to convince yourself that the answer is as given above, namely the exponential of the second-order term. Give a simple reason that all the odd-order terms in J vanish.

(c) Now consider the particular choice of source

$$J_R(x) = \theta(T - t)\theta(t + T)\left(q_1\delta^{D-1}(x) + q_2\delta^{D-1}(x - R)\right)$$

where $R^\mu = (0, \vec{R})^\mu$ is a constant vector and $T \gg R \gg 1/m$. This corresponds to two static, localized source particles interacting for a time duration T, at a separation $\vec{R}$. Show that in the regime indicated, the answer has the form

$$Z[J_R] \propto e^{-\mathbf{i}TV(R)}$$

where for $D = 3+1$, $V(R) \sim q_1q_2\frac{e^{-mR}}{R}$ is the Yukawa potential. (We are interested in the dependence on R, and therefore can ignore factors in Z that don't depend on R.)

Interpret V as the potential energy of interaction between the two external particles. To obtain the force between the sources mediated by ϕ, differentiate the potential with respect to the displacement: $F = -\partial_R V$. Is the force attractive or repulsive when $q_{1,2}$ have the same sign?

2. **Schwinger-Dyson equations**

Consider the path integral

$$\int [D\phi]e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi]\frac{\delta}{\delta\phi(x)}\,(\text{anything})$$

(as long as 'anything' doesn't grow at large ϕ). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int[D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial\phi_x}\,(\text{anything})$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and are called Schwinger-Dyson equations. (Be warned that these equations are sometimes also called Ward identities.) They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \left(\phi(y) e^{\mathbf{i}S[\phi]} \right)$$

to conclude that

$$\langle \mathcal{T} \frac{\delta S}{\delta\phi(x)} \phi(y) \rangle = -\mathbf{i}\delta(x-y). \tag{1}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (3) for the case of a free massive scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\langle \mathcal{T} \left(+\Box_x + m^2 \right) \phi(x)\phi(y) \rangle = -\mathbf{i}\delta(x-y), \tag{2}$$

with $\Box_x \equiv \partial_{x^\mu}\partial^{x^\mu}$.

(c) Find the generalization of (4) satisfied by (time-ordered) three-point functions of the free field ϕ.

(d) Derive the equation (3) (for a free theory) from a more canonical (*i.e.* Hamiltonian) point of view, by considering what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x} - \vec{y}).$$

Do not neglect the fact that $\partial_t \theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]

3. **Making connected functions out of 1PI functions.**

In this problem we prove the relations illustrated in Fig. 1. More precisely, we prove that the ϕ^n term in the expansion of $\Gamma[\phi]$ is the 1PI n-point function.

(a) Show using the definitions of ϕ_c and the 1PI effective action that

$$\frac{\delta^2 W}{\delta J(x)\delta J(y)} = \frac{\delta\phi(y)}{\delta J(x)} = \left(\frac{\delta J(x)}{\delta\phi(y)} \right)^{-1} = -\left(\frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(y)} \right)^{-1}. \tag{3}$$
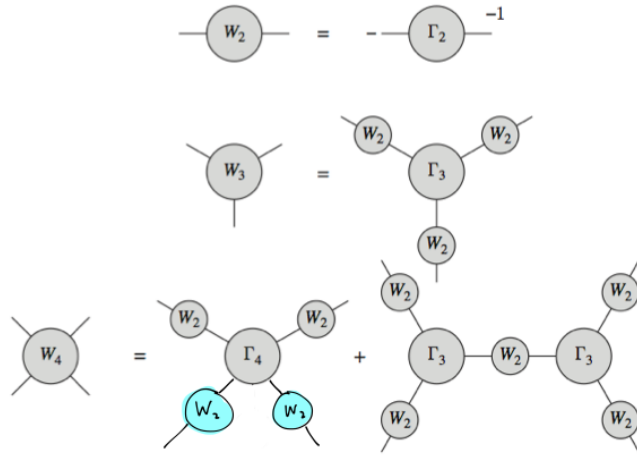
(where $\phi \equiv \phi_c$ here). Inverse here means in the sense of integral operators: $\int d^D z K(x,z) K^{-1}(z,y) = \delta^D(x-y)$. Convince yourself that (5) can be written more compactly as:

$$W_2 = -\Gamma_2^{-1}.$$

(b) Show that

$$W_3(x,y,z) = \int dw_1 \int dw_2 \int dw_3 W_2(x,w_1) W_2(y,w_2) W_2(z,w_3) \Gamma_3(w_1,w_2,w_3) .$$

Do this by differentiating (5) again with respect to J and using the matrix differentiation formula $dK^{-1} = -K^{-1}dK K^{-1}$ and the chain rule.

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Legendre transform $W[J] = \Gamma[\phi] + \int \phi J$ makes trees.

Figure 1: [From Banks, *Modern Quantum Field Theory*, with some improvement] W_n denotes the connected n-point function, $\left(\frac{\partial}{\partial J}\right)^n W[J] = \langle \phi^n \rangle$. Γ_n denotes the 1PI n-point function, as encoded in the 1PI effective action.

(c) Show that W_n can be constructed from Γ and W_2 as indicated in the figure. The idea is to compute W_n by taking more derivatives with respect to J. To get the rest of the W_n requires an induction step.