University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Field Theory C (215C) Spring 2013
Assignment 4

Posted May 1, 2013 **Due 11am, Thursday, May 16, 2013**

Problem Set 4

1. **Spectral representation at finite temperature.**

   In lecture we (will) have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

   $$\mathbf{i}\mathcal{D}(x) = \langle 0|\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

   Derive a spectral representation for the two-point function of a scalar operator at a nonzero temperature:

   $$\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta\mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n}\langle n|\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)|n\rangle.$$

   Here $Z_\beta \equiv \mathrm{tr} e^{-\beta\mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result.

2. **Practice with cutting rules.**

   [roughly, this is Zee problem III.8.2] Consider the massive scalar field theory in four dimensions with $\frac{g}{3!}\phi^3$ interaction. Show that to order g^4 there is a 'box diagram' contributing to $\phi\phi \to \phi\phi$ scattering with amplitude

   $$\mathcal{I} = g^4 \int \frac{đ^4 k}{(k^2 - m^2)((k+p_2)^2 - m^2)((k-p_1)^2 - m^2)((k+p_2-p_3)^2 - m^2)}$$

   (where m^2 means $m^2 - \mathbf{i}\epsilon$).

   Here the momenta of the ϕ particles are specified to be $p_1 + p_2 \to p_3 + p_4$. Calculate the integral explicitly as a function of $s = (p_1+p_2)^2$ and $t = (p_3 - p_2)^2$. Study the analyticity properties of $\mathcal{I}(s)$ at fixed t – there should be a cut starting at $s = (2m)^2$. Evaluate the discontinuity of $\mathcal{I}$ across the cut (recall from complex analysis that for a cut of a real function on the real axis, this discontinuity is directly proportional to the imaginary part of the function) and verify Cutkosky's cutting rule. Check that optical theorem works.

   Bonus material: Think about the dependence of $\mathcal{I}(t)$ on t at fixed s.

In the next problems, we will study free massless bosons in two dimensions. This system has many physical applications – *e.g.* in string theory, and at the edge of quantum Hall systems. It is an example of a *conformal field theory*. It is a field theory of the kind I have advertised several times, where the excitations are *not particles*.

3. **There are no Goldstone bosons in two dimensions.**

   [Perhaps this is more of a diatribe than a problem.]

   (a) Consider a massless scalar X in 2d, with action

   $$S[X] = -\frac{1}{4\pi}\int d^2\sigma \partial_a X \partial^a X. \tag{1}$$

   Show that the euclidean Green function G_2 satisfies

   $$\nabla^2 G_2(z, z') = -2\pi\delta^2(z - z') \tag{2}$$

   ($z = \sigma_1^E + \mathbf{i}\sigma_2^E$)[1] and is given by

   $$G_2(z, z') = \frac{1}{2\pi}\ln|z - z'|,$$

   for example by Fourier transform.

   (b) The long-distance behavior of G_2 has important implications for the physics of massless scalars in two dimensions. Thinking of G_2 as the two point function of the massless scalar

   $$G_2(z, z') = \langle X(z)X(z')\rangle$$

   let's ask the following question:

   There is no potential energy for the field X in (1). Someone used to physics in 3+1 dimensions might think that this means that there is a vacuum for every value of X. Let's try to fix the expectation value of the scalar $\langle X\rangle = x$ and see what happens. Perturb the putative vacuum $|x\rangle$ a little bit at the position z by inserting the operator X there. To measure what happens, insert the operator X at z'. The correlator G_2 can thus be interpreted as a measurement of how the effects of our perturbation fall off with distance. What happens? Contrast this with the behavior you would see for a scalar field with a flat potential in more than two dimensions (recall problem 1 of problem set 2). Note that the case of (0+1) dimensional QFT (*i.e.* quantum mechanics) is even more problematic in the infrared.

   One way to arrive at an action like (1) is if the field X arises as a Goldstone boson associated with a symmetry $X \to X + a$, which would be broken by fixing the vacuum $|x\rangle$. Then it is guaranteed by Goldstone's theorem that the action can only depend on derivatives of X.

   But note that the Goldstone-ness of the massless bosons (*i.e.* whether they are massless because of a broken symmetry) is not crucial for this discussion. In non-supersymmetric

[1] Recall Schwinger-Dyson equations from problem set 2.

sigma models, one expects massless bosons whose masslessness is not protected by a symmetry to be lifted quantumly, but in supersymmetric theories, this need not be the case, and different points on the space of minima of the potential need not be related by a symmetry.

This result is called the Coleman-Mermin-Wagner (sometimes Hohenberg, too) Theorem. Coleman's paper on the subject is S. Coleman, "There are no Goldstone bosons in two-dimensions," *Commun. Math. Phys.* **31**:259-264 (1973).

4. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

Consider a collection of n two-dimensional free bosons X^μ governed by the action

$$S = -\frac{1}{4\pi g}\int d^2\sigma \partial_a X_\mu \partial^a X^\mu .$$

[The coupling g can be absorbed into the definition of X if we prefer, but it is useful to leave this coupling constant arbitrary since different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer.]

Until further notice, we will assume that X takes values on the real line.

(a) Rotate $e^{\mathbf{i}S}$ to Euclidean space ($d^2\sigma = -\mathbf{i}(d^2\sigma)_E$) and compute the Euclidean generating functional

$$Z[J] = \langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \rangle \equiv Z_0^{-1}\int [dX] e^{\mathbf{i}S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

(where $Z_0^{-1} \equiv Z[J=0]$ but please don't worry too much about the normalization of the path integral).

[Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

[Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

[Cultural remark 1: this field theory describes the propagation of featureless strings in n-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point σ is mapped.

Cultural remark 2: this is an example of a *conformal field theory.* In particular recall that massless scalars in $D = 2$ have engineering dimension zero.]

(b) Show that

$$\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'}k_i \cdot X(\sigma^{(i)})} : \rangle = \delta^n\left(\sum_i k_i^\mu\right) \prod_{i,j=1}^{N} |z_i - z_j|^{-\alpha' g k_i \cdot k_j} \tag{3}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, α' is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and k_i^μ are a set of arbitrary n-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the the delta function in k.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv : e^{\mathbf{i}aX} :$ has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\langle \mathcal{O}_a(z) \mathcal{O}_b^\dagger(0) \rangle = \delta(a-b) \frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n = 1$) X which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R \ .$$

What values of a label single-valued operators $: e^{\mathbf{i}aX} :$? How should we modify (3)?