University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Field Theory C (215C) Spring 2013
(Bonus) Assignment 7

Posted June 4, 2013 **Due 11am, Thursday, June 13, 2013**

Problem Set 7

1. **Large-N saddle points** [extra credit] [from Halpern]

   Consider the partition function for an N-vector of sscalar fields in D dimensions

$$Z = \int [D\phi] e^{\mathbf{i}S[\phi]}, \quad S[\vec{\phi}] = \int \mathrm{d}^D x \left(\frac{1}{2} \partial\phi^a \partial\phi^a - NV\left(\frac{\vec{\phi}^2}{N}\right)\right)$$

   with a general 2-derivative $O(N)$-invariant action. We're going to do this path integral by saddle point, which is a good idea at large N. As usual, the constant prefactors in Z drop out of physical ratios so you should ignore them.

   (a) Change variables to the $O(N)$ singlet field $\zeta \equiv \vec{\phi}^2/N$ by inserting the identity

$$1 = \int [D\zeta] \delta\left[\zeta - \frac{\vec{\phi}^2}{N}\right]$$

   into the path integral representation for Z. Represent the functional delta function as

$$\delta\left[\zeta - \frac{\vec{\phi}^2}{N}\right] = \int [D\sigma] e^{\mathbf{i}\int \mathrm{d}^D x \sigma\left(\vec{\phi}^2 - \zeta N\right)}.$$

   Do the integral over ϕ^a to obtain

$$Z = \int [D\zeta D\sigma] e^{\mathbf{i}N S_{\text{eff}}[\zeta,\sigma]}.$$

   Determine $S_{\text{eff}}[\zeta, \sigma]$.

   (b) The integrals over ζ, σ have a well-peaked saddle point at large N. Obtain the coupled large-N saddle point equations for the saddle point configurations ζ_0, σ_0, and in particular the equation

$$\zeta_0(x) = \left(\frac{\mathbf{i}}{-\Box - 2V'(\zeta_0)}\right)_{xx}$$

   (the subscript denotes a matrix element of the position-space operator).

(c) [more optional] Show that

$$\frac{\delta}{\delta\sigma(x)}\mathrm{tr}\log\left(-\Box+\sigma\right) = \left(\frac{1}{-\Box+\sigma}\right)_{xx}$$

by Taylor expansion.

(d) At large N, we know that

$$\zeta_0(x) \overset{N\to\infty}{=} \langle\frac{\vec{\phi}^2(x)}{N}\rangle = \zeta_0, \text{ constant.}$$

Use this to show that the saddle point equation is the *gap equation*

$$\zeta_0 = \int đ^D k_E \frac{1}{k_E^2 + 2V'(\zeta_0)}$$

which determines ζ_0, the expectation value of the order parameter $\langle\vec{\phi}^2/N\rangle$.

(e) What class of diagrams did you just sum?

(f) **Toy model emphasizing the similarity to our discussion of BCS.** You might be worried that in the discussion of BCS I didn't introduce the analog of ζ (which is the actual pair field). In this small, very optional, sub-problem I want to show that what we did there is the same as what we did here. Consider the integral

$$Z = \int \mathrm{d}^2x\ e^{-a|x|^2 - u|x|^4};$$

here x is a proxy for the fermion field ψ, but it is just a complex number. Now we insert

$$1 = \int \mathrm{d}^2\zeta\ \delta[\zeta - x^2]\delta[\bar{\zeta} - \bar{x}^2] = \int \mathrm{d}^2\zeta \int \mathrm{d}^2\sigma\ e^{\mathbf{i}\bar{\sigma}\left(\zeta - x^2\right) + \mathbf{i}\sigma\left(\bar{\zeta} - \bar{x}^2\right)}$$

to find

$$Z = \int \mathrm{d}^2\zeta \int \mathrm{d}^2\sigma e^{-u|\zeta|^2 + \mathbf{i}\left(\sigma\bar{\zeta} + \bar{\sigma}\zeta\right)} \int \mathrm{d}^2x\ e^{-a|x|^2 - \mathbf{i}\bar{\sigma}x^2 - \mathbf{i}\sigma\bar{x}^2}$$

Do the x integral and find

$$Z = \int \mathrm{d}^2\zeta \int \mathrm{d}^2\sigma e^{-u|\zeta|^2 + \mathbf{i}\left(\sigma\bar{\zeta} + \bar{\sigma}\zeta\right) - \log\left(a^2 - |\sigma|^2\right)}.$$

Now we can do the ζ integral to get

$$Z = \int \mathrm{d}^2\sigma e^{-\frac{1}{u}|\sigma|^2 - \log\left(a^2 - |\sigma|^2\right)}$$

which is what we would have found using the usual HS trick as described in lecture. The relationship between ζ and σ on-shell (*i.e.* at the saddle point of the (gaussian) ζ integral) is $\zeta \sim \mathbf{i}\sigma$, but really they are conjugate variables. Now do the σ integral by saddle point to find the analog of the gap equation for this 0+0 dimensional QFT.

(g) [super-optional] Show that the BCS interaction is a marginally relevant perturbation of the Fermi liquid fixed point(s). (For help, see the papers by Polchinski or by Shankar cited in the lecture notes.)

2. **Chiral anomaly in two dimensions.** [extra credit]

Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

$$S = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\left(\mathbf{i}\gamma^\mu\left(\partial_\mu + eA_\mu\right) - m\right)\psi.$$

By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j^5_\mu \equiv \mathbf{i}\bar{\psi}\gamma^m u\gamma^5\psi$ is

$$\partial_\mu j^5_\mu = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

3. **Topological terms in QM** [extra credit] [from Abanov]

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B}\cdot \mathrm{d}\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta \mathrm{d}\tau\left(\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi}\right)}.$$

Here

$$\phi \equiv \phi + 2\pi \tag{1}$$

is a coordinate on the ring. Because of the identification (1), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta}Q\tau + \sum_{\ell\in\mathbb{Z}}\phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{2}$$

(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}tn^2 + \mathbf{i}zn} = \sqrt{\frac{2\pi}{t}}\sum_{\ell\in\mathbb{Z}} e^{-\frac{1}{2t}(z-2\pi\ell)^2}. \quad]$$

(c) Use the result from the previous part to determine the energy spectrum as a function of θ.

(d) [more optional] Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta-\pi}{M} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

The lesson here is that (even in this simple QM example) total derivative terms in the action do affect the physics.

4. **Geometric Quantization of the 2-torus.** [extra credit]

Redo the analysis that we did in lecture for the two-sphere for the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x+2\pi, y+2\pi)$; use $N\mathrm{d}x\wedge\mathrm{d}y$ as the symplectic form. Show that the resulting Hilbert space represents the heisenberg algebra

$$e^{\mathbf{ix}}e^{\mathbf{iy}} = e^{\mathbf{iy}}e^{\mathbf{ix}}e^{\frac{2\pi\mathbf{i}}{N}}.$$

(I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with N sites.

5. **Particle on a sphere with a monopole inside.** [extra credit]

Consider a particle of mass m and electric charge e with action

$$S[\vec{x}] = \int \mathrm{d}t \left(\frac{1}{2}m\dot{\vec{x}}^2 + e\dot{\vec{x}} \cdot \vec{A}(\vec{x})\right)$$

constrained to move on a two sphere of radius r in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a *magnetic monopole* inside this sphere: this means that $4\pi g = \int_{S^2} \vec{B} \cdot \mathrm{d}\vec{A} = \int_{S^2} F$, where $F = \mathrm{d}A$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)

(a) Find an expression for $A = A_i\mathrm{d}x^i = A_\theta\mathrm{d}\theta + A_\varphi\mathrm{d}\varphi$ such that $F = \mathrm{d}A$ has flux $4\pi g$ through the sphere.

(b) Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.

(c) Take the limit $m \to 0$. Count the states in the lowest Landau level. It should agree with the result from lecture.

6. **Coherent state quantization** [extra credit]

(a) Start with first order action $S = \int \mathrm{d}t\ z_\alpha^\dagger \dot{z}_\alpha$. Show that the Hamiltonian is $\mathbf{H} = 0$.

(b) Check the completeness relation in the spin 1/2 coherent state basis.

(c) Show that different spinor representations, *i.e.* different choices of ψ in

$$z = \begin{pmatrix} e^{\mathbf{i}(\psi+\varphi/2)}\cos\theta/2 \\ e^{\mathbf{i}(\psi-\varphi/2)}\sin\theta/2 \end{pmatrix}$$

shift the coefficient of the total derivative $\dot{\varphi}$ part of the WZW functional.