Casimir forces between gently curved surfaces

David Leon[1]

[1]Department of Physics, University of California at San Diego, La Jolla, CA 92093

A brief look at the Casimir force, from a simplified toy model to a path integral approach incorporating small deviations from flatness.

INTRODUCTION

One of the consequences of our quantum mechanical world is that the vacuum is not really empty. Fluctuations of quantum fields contribute to a nonzero vacuum energy, and while we can safely ignore this divergent energy in most calculations since we only care about relative energy scales, we can agree that changes in this vacuum energy should certainly be observable.

Consider for example two parallel conducting plates. We know that conductors have no internal electric field which imposes the condition that the modes between the two plates must vanish at the boundaries. Since the energy of these modes is dependent on the plate separation, the vacuum energy will change with separation and where we have a spatially varying energy there is a force proportional to its gradient.

This force has been experimentally verified at small separations and simple conductor geometries but it is difficult to derive an expression for the force between more complicated curved surfaces. We will first get acquainted with the Casimir force in a simple case of perfectly flat plates and then attempt to understand the case for gently varying conductor surfaces in the path integral formulation.

A SIMPLIFIED MODEL

As is the case for most instructive examples we can make a number of simplifying assumptions to calculate the Casimir force between two plates. First, rather than dealing with the technical complications of the electromagnetic field let us simply use a real scalar field. As stated before, we will assume the plates are flat and perfect conductors. Finally, let us restrict the problem to one spatial dimension for simplicity.

One further note: let us consider three parallel plates where the two outer plates are stationary and we vary the position of the innermost plate. This will allow us to account for the change in the modes on either side of the moving plate. To find the force between two plates, we will take the separation between plates 1 and 2 to be much smaller than the distance between plates 2 and 3.

Now recall that the ground state energy of a free scalar field is given by:

$$\langle 0|H|0\rangle = V \int đ^D k \frac{1}{2}\hbar\omega_k$$

For our parallel plate setup, the contributions from between two plates only allows discrete energy values giving an energy per unit area of

$$E = \sum_n \frac{1}{2}\hbar\omega_n$$

With frequency $\omega_n = n\pi/d$ for a plate separation d. Of course this sum is still divergent as it includes arbitrarily high energies so let's regularize the sum with an ultraviolet cutoff a^-1 so that the energy is finite and in the limit $a->0$ we can recover the cutoff independent result.

$$\begin{aligned} E &= \frac{\pi}{2d}\sum_n n e^{-an\pi/d} = -\frac{1}{2}\frac{\partial}{\partial a}\sum_n e^{-an\pi/d} \\ &= -\frac{1}{2}\frac{\partial}{\partial a}\frac{1}{1-e^{-a\pi/d}} = \frac{\pi}{2d}\frac{e^{a\pi/d}}{(e^{a\pi/d}-1)^2} \end{aligned}$$

Since we have a large cutoff, a is small and we can expand the exponentials as $1 + a\pi/d$ to arrive at:

$$E = \frac{d}{2\pi a^2} - \frac{\pi}{24d}$$

Now in our setup we have three plates, two outer plates are fixed at some very large separation L while the inner plate is some small distance d from one of the plates (and a distance $L-d$ from the other). The contribution to the energy from these two plate separations is:

$$\begin{aligned} E(d) + E(L-d) &= \left(\frac{d}{2\pi a^2} - \frac{\pi}{24d}\right) + \left(\frac{L-d}{2\pi a^2} - \frac{\pi}{24(L-d)}\right) \\ &= \frac{L}{2\pi a^2} - \frac{\pi}{24d} - \frac{\pi}{24(L-d)} \end{aligned}$$

We notice that the term containing the cutoff dependence is a constant so when we find the force from a variation of d we arrive at a cutoff independent term:

$$F = \frac{\partial(E(d) + E(L-d))}{\partial d} = \frac{\pi}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right)$$

Which in the limit $d << L$ results in:

$$F = \frac{\pi}{24d^2}$$

So we arrive at the fact that from the divergent vacuum energy we've ignored for so long in our study of quantum fields yields a finite measurable force.

To make the connection to three spatial dimensions, if we perform the calculation in D spatial dimensions it can be shown that the d-dependent part of the vacuum energy is proportional to $d^{-(D)}$ so that the Casimir force scales as $F \propto d^{-(D+1)}$

In the spirit of Ph 215c we can also make the observation that in this case we obtain a cutoff independent result where in reality conducting plates have a natural frequency cutoff due to their finite conductivites, so arriving at a cutoff independent result is in some ways deceiving.

Now that we've gone through a simple (and hopefully intuitive) approach let us move on to a path integral approach and see if we can move beyond perfectly flat plates.

PATH INTEGRAL APPROACH

Again for simplicity let us consider the a scalar field.

In order to incorporate our two conducting plates into the path integral we can insert delta functions of the scalar field along the planes of our conducting plates. So for example in three spatial dimensions with two plates we can define $z_1 = z_1(x_1, y_1)$ and $z_2 = z_2(x_2, y_2)$ as the profiles of our two conducting plates which can have small deviations from complete flatness. Upon integrating over ϕ the delta functions will impose the condition that the field vanishes at the plates. The Euclidean path integral will be:

$$Z = \int [D\phi]\delta_1(\phi)\delta_2(\phi)e^{-S_0[\phi]}$$

Where the delta functions can be expanded in terms of an auxiliary field λ:

$$\delta_i(\phi) = \int [D\lambda]e^{-i\int d^4x\phi(x)\lambda(x_\parallel)\delta(z-z_i(x_\parallel))}$$

The $x_\parallel$ denotes that the auxiliary field is only defined on the surface corresponding to the plates. Essentially this is the familiar procedure of adding a source term for ϕ, so we can now integrate out ϕ and arrive at an integral that is quadratic in the source term:

$$Z = \int [D\lambda] \exp\left(-\frac{1}{2}\int d^3x_\parallel d^3y_\parallel \lambda_a(x_\parallel)G_{ab}(x_\parallel, y_\parallel)\lambda_b(y_\parallel)\right)$$

The indices a, b run from 1 to 2, corresponding to the two conducting plates, and the field is defined across the surface of these plates (λ_1 along z_1 and λ_2 along z_2). As this is quadratic in λ we know the result will be (up to some normalization) a functional determinant of G which gives us:

$$\Gamma = -\log(Z) = -\log(\mathrm{Det}(G)^{-1/2}) = \frac{1}{2}\mathrm{Tr}\,[\log(G_{ab}(x_\parallel, y_\parallel))]$$
$$= \int d^3x_\parallel d^3y_\parallel \frac{1}{2}\mathrm{tr}[\log(G_{ab}(x_\parallel, y_\parallel))]$$

So we have a nice result in terms of the scalar propagator and the surfaces of the conducting plates. The procedure at this point is to first note that we are still modeling plates that are close and approximately flat, so we can separate the surface profiles into an average value and a small deviation $z = d + \eta(x_\parallel)$ which allows for an expansion of Γ in powers of η. This condition allows us to find the leading order correction to the Casimir energy of order η^2 so that the energy between two approximately parallel plates with small deviations according to $z = d + \eta(x_\parallel)$ is roughly of the form:

$$E(d, \eta) = U(d) + \int đ^2k G(d,k)|\eta(k)|^2$$

Where U is the result from using flat plates. Expanding G at low energy gives a k^2 term for the final result that the leading order correction to the Casimir energy is proportional to $|\nabla\eta|^2$.

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