Final reading report: BPHZ Theorem

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This reports summarizes author's reading on BPHZ theorem, which states that divergences in every Feynman diagram can be eliminated for both renormalizable and non-renormalizable quantum field theories.

INTRODUCTION

BPHZ theorem gives an affirmative answer to removal of divergences to all orders in diagrammatic calculations in quantum field theories. This report summarizes author's reading on Bogoliubov and Parasiuk's (BP) [1] work (commented in Peskin's book as "begun"), and Hepp's work (H) [2] (as "completed"). While Zimmerman's (Z) [3] paper (as "elegantly refined") is not included.

ORIGIN OF DIVERGENCE AND CONCEPT OF GENERALIZED FUNCTIONS

Feynman propagator for scalar field has the following types of singularities on the light cones: $\delta(\lambda)$, $\frac{1}{\lambda}$, $\theta(\lambda)$, $\ln|\lambda|$, where $\lambda = \sqrt{|x^\mu x_\mu|}$. The exact meaning of these expressions is clarified in the theories of generalized functions, which view them as linear functionals on some test function space. Yet in interacting theories, when singularities of individual propagators superimpose on one another, the result is ill-defined even in the sense of generalized functions. Then a certain subtraction method is needed to yield divergence-free results. And Hepp noted that "the renormalization theory of Dyson is in this framework a constructive form of the Hahn-Banach theorem".

ARBITRARINESS IN S-MATRIX

Consider interaction picture. Bogoliubov argues in his book that if we temporarily set aside Hamiltonian formalism, we will find that there is arbitrariness in S-matrix if the only requirements are Lorentz covariance, unitarity and causality. For convenience, we introduce "switching-on" intensity $g(x)$, $0 \leq g(x) \leq 1$, and then S-matrix becomes a functional of $g(x)$. The final result can be obtained by setting $g(x) = 1$. The three conditions can be expressed as,

1. covariance: $S(Lg)U_L = U_L S(g)$, in which $Lg(x) = g(L^{-1}x)$, U_L is the representation of corresponding Lorentz transformation on Hilbert space;
2. unitarity: $S(g)^\dagger S(g) = I$;
3. locality: $S(g_1 + g_2) = S(g_2)S(g_1)$, $g_i \neq 0$ only on G_i, in which $G_2 > G_1$ means all points of region G_2 occur at times later than all points of G_1.

Expand $S(g)$ as

$$S(g) = 1 + \sum_{n\geq 1} \frac{1}{n!} \int S_n(x_1, ..., x_n) g(x_1) g(x_2) ... g(x_n) dx_1 dx_2 ... dx_n \tag{1}$$

, and impose the three requirements, one finds that S_n is determined inductively by S_1 to S_{n-1} up to a so called "quasi local" operator

$$i\Lambda_n(x_1, ..., x_n) = Z(... \frac{\partial}{\partial x_i} ...) \delta(x_1 - x_2) ... \delta(x_1 - x_n) \, , \tag{2}$$

where Z is a polynomial, which only does not vanish when all points coincide. It is surprising that these additional terms in S_n can be equivalently incorporated into Lagrangian as

$$S(g) = T(exp(i \int L(x, g) dx)) \tag{3}$$

$$L(x;g) = L(x)g(x) + \sum_{\nu\geq 2} \frac{1}{\nu!} \int \Lambda_\nu(x, x_1, ..., x_{\nu-1}) g(x) g(x_1)...g(x_{\nu-1}) dx_1...dx_{\nu-1} \, . \tag{4}$$

. From (2) one sees that every single term in the summation is local. Expand S(g) out one finds the expression which will be useful in later discussions of R-operation:

$$S_n(x) = \sum_{1\leq m\leq n} i^m P(x_1, ...x_{\nu_1}|x_{\nu_1+1}...|..., x_n) \times T[\Lambda_{\nu_1}(x_1, ...x_{\nu_1})...\Lambda_{\nu_m}(...x_n)] \, , \tag{5}$$

where we have denoted $L(x)$ as $\Lambda_1(x)$, and P denotes symmetrization of all x's modulus internal permutations of m groups of x's.

BOGOLIUBOV R-OPERATION

Definition of coefficient function

Consider an aribitrary operator as a product of field operators. We may normal order it via Wick's theorem. Coefficient function corresponding to a certain diagram G is defined to be the c-function determined by G via Feynman rules, which appear as coefficient of the normal ordered field operators.

α-representation of coefficient functions and appearance of ultra-violet divergence at $\alpha = 0$

We write Feynman propagator as

$$reg(D(p)) = Z(-i\nabla_q)|_{q=0} i \int_0^\infty e^{i\alpha(p^2-m^2+i\epsilon)+ip\cdot q} I(\alpha) d\alpha \, , \tag{6}$$

in which "reg" means "Pauli Villars regularized", $I(\alpha)$ is a factor resulting from Pauli-Villars regularization which is essentially not important to our discussion, and $Z(p)$ is a polynomial which is nontrivial for non-scalar fields. For a certain Feynman diagram, it is important that written in in this form we can perform the integral by momentum delta function and Gaussian integral over remaining loop momentums. We are left only with external momentum, and the result for scalar field (Z=1) is as follows

$$\int exp(i \sum \alpha_l p_l^2 + i \sum p_l q_l \prod \delta(\sum p + k_q) \prod dp_l) = \tag{7}$$

$$\frac{i^{n+1}\delta(\sum k)}{\pi^{2n-2L-2}} \int_0^\infty d\alpha_1...d\alpha_L D^{-2}(...\alpha...) \times \exp(i \frac{Q(...\alpha;...k)}{D(...\alpha)} - i \sum_l \alpha_l (m_l^2 - i\epsilon)) \prod_{1\leq l\leq L} I(\alpha_L) \, , \tag{8}$$

in which k is external momentum attached to some vertex $i \, , 1 \leq i \leq n$, l denotes an internal line. D is the determinant of the symmetric bilinear form in the quadratic term of loop momentums, D^{-1} in the exponential is from inverse of the bilinear form and the algebraic cofactors are included in Q. That D^{-2} rather than $D^{-1/2}$ appears outside the exponential is because the momentum has four components. The exact form of D and Q can be obtained (for a graphic expression of D and Q see "29.5 structure of the exponential quafratic form" in Bogoliubov's book), but to our purpose, we only need the fact that D is a homogenous polynomial in α with degree $L-n+1$, Q is a homogeneous polynomial of α with degree $L - n + 2$, and quadratic in k. Integral of large α is suppressed by ϵ term and due to homogeneity of D the only possible divergence appears at $\alpha = 0$ end.

Definition of $\Delta-$ and $M-$ operation

Suppose we want to get coefficient function corresponding to diagram G of operator $T(\Delta_\nu(x_1...x_\nu)L(x_{\nu+1})...L(x_n))$. We can simply remove all propagators inside sudiagram G_ν of the coefficient function of $T(L(x_1)L(x_2)...L(x_n))$ and replace it with the coefficient function $d_G = CF(\Lambda_\nu, G)$ of $\Lambda_\nu(x_1, ..., x_\nu)$ for the diagram G_ν, while keeping propagators connecting G_ν to outside unchanged. Such an operation is called $\Delta(G_\nu)$-operation. In this way the recursive expression for $S_n(x_1, ..., x_n)$ in part 3 can be written in the form of coefficient function as

$$CF(S_n, G) = \{\sum_{1\leq m\leq n} \Delta(G_1)...\Delta(G_2)\}CF(T(L(x_1)...L(x_n)), G) \tag{9}$$

in which CF means coefficient function and sets of vertices of G_i constitute a partition of those in G. Bogoliubov R-operation is defined as

$$\begin{aligned} R(G) &= \sum_{1\leq m\leq n} \Delta(G_1)...\Delta(G_2) \\ &= 1 + \sum_{2\leq m\leq n-1} \Delta(G_1)...\Delta(G_2) + \Delta(G)\,. \end{aligned} \tag{10}$$

Since our current goal is to just get finite result, we can define $\Delta(G)$ to be nonzero only on 1PI diagrams. For disconnected and 1PR diagrams, the convergence of the whole diagram is ensured by the convergence of its disconnected or 1P-removed disconnected subparts, and so there is no need to modify G as a whole.

Explicit form of coefficient function d_G

It is not hard to see from α - representation that every differentiation with respect to external momentum lowers the pole order at $\alpha = 0$ by $\frac{1}{2}$. This motivates Bogoliubov and Parasiuk to define $\Delta(G)$ inductively as follows: $\Delta(G) = -M(G)\sum_{2\leq m\leq n}\Delta(G_1)...\Delta(G_m)$, in which the effect of $M(G)$ on a function of the form $\delta(\sum k)F(k)$ is

$$M(G)[\delta(\sum k)F(k)] = \delta(\sum k)\{F(k)\}_{\omega(G)}\,, \tag{11}$$

where $\{F(k)\}_{\omega(G)}$ is the Maclaurin's expansion of $F(k)$, and $\omega(G)$ is the superficial degree of divergence of 1PI diagram G. Thus by definition of $R(G)$ we get

$$\begin{aligned} R(G) &= \sum_{2\leq m\leq n} \Delta(G_1)...\Delta(G_m) + \Delta(G) \\ &= (1 - M(G)) \sum_{2\leq m\leq n} \Delta(G_1)...\Delta(G_m)\,. \end{aligned} \tag{12}$$

An example of R-operation is included in the Appendix. Intuitively, $\Delta(G)$ removes the divergence when all α's approach zero simultaneously, while other $\Delta(G_i)$'s compensate singularities in regions where only some of the α's go to zero. R-opertion can remove all divergences from the original expansion.

Rigorous proof of removal of all divergences by R-operation

Bogoliubov's book does not provide a rigorous proof for this. A first complete proof was given by Klaus Hepp, whose paper to me is very technical and hard to read, and I have not finished completely yet. But the rough idea is the following. Hepp observed that one can prove convergence of some partial sums in R-operated diagram in the region $\alpha_{l_1} \geq \alpha_{l_2} \geq ... \geq \alpha_{l_L}$. Based on this he developed a complicated method to decompose the original sum into a whole bunch of sums, in each of which the integral is convergent, and thus proved the convergence probelm. The decompostion is in Lemma 2.4 in his paper, which I will not replicate due to limitation of space.

CLASSIFICATION OF FIELD THEORIES

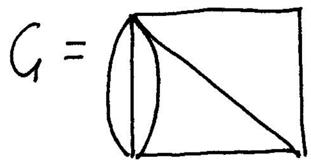
Recall that every coefficient function is multiplied to a normal ordered product of field operators, which are actually attached to external legs. Plugging d_G back into $\Lambda_n(x_1...x_n)$ in the Lagrangian, one finds that in $i\Lambda_n(x_1,...x_n) = Z(...\frac{\partial}{\partial x_i}...)\delta(x_1-x_2)...\delta(x_1-x_n)$ all delta functions make the term local while derivatives in Z are converted onto derivatives on field operators after integration by part. It is natural to define two counter-terms to be of the same type if they have the same field operators and derivatives, and so we can merge them together in the Lagrangian. If finally we have a finite number of different types of counter-terms, then such a theory is called renormalizable while it is called non-renormalizable if there is an infinite number of different types of counter-terms. Yet regardless of renormalizability, one can always eliminate divergences in every single Feynman diagram by Bogoliubov R-operation. One can view performance of counter terms as follows: break counter terms to a sum of coefficient functions multiplied with normal ordered product of field operators, and then insert them into appropriate positions in Feynman diagrams, which is essentially Δ-operation. Then divergences are removed.

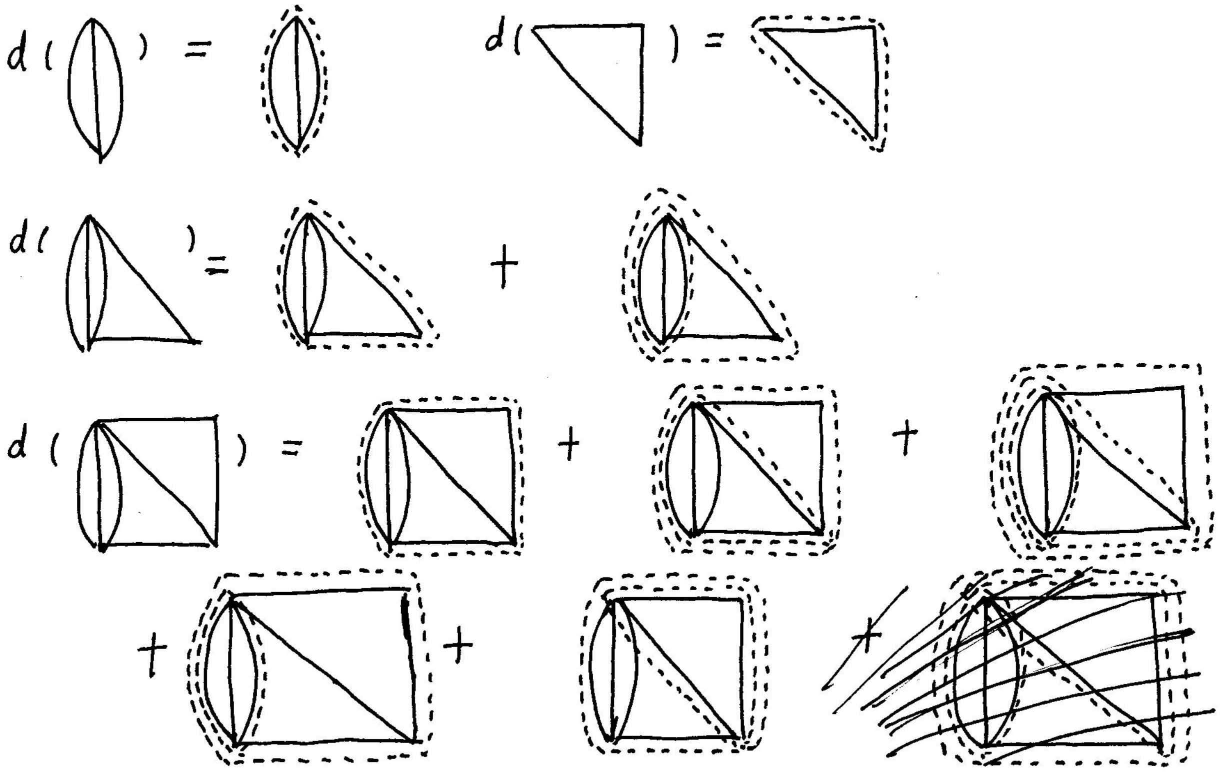
DETERMINATION OF COUNTER-TERMS VIA RENORMALIZATION CONDITION

It is obvious that there is a huge arbitrariness of choice of R-operation if we just want to make the final result finite (for instance add terms not affecting pole structure into definition of M-operation). A natural question is how to determine the form of R-operation on physical consideration. This seems to be disscussed in chapter 6 of Bogoliubov's book, but unfortunately I have not started reading that part due to limitation of time.

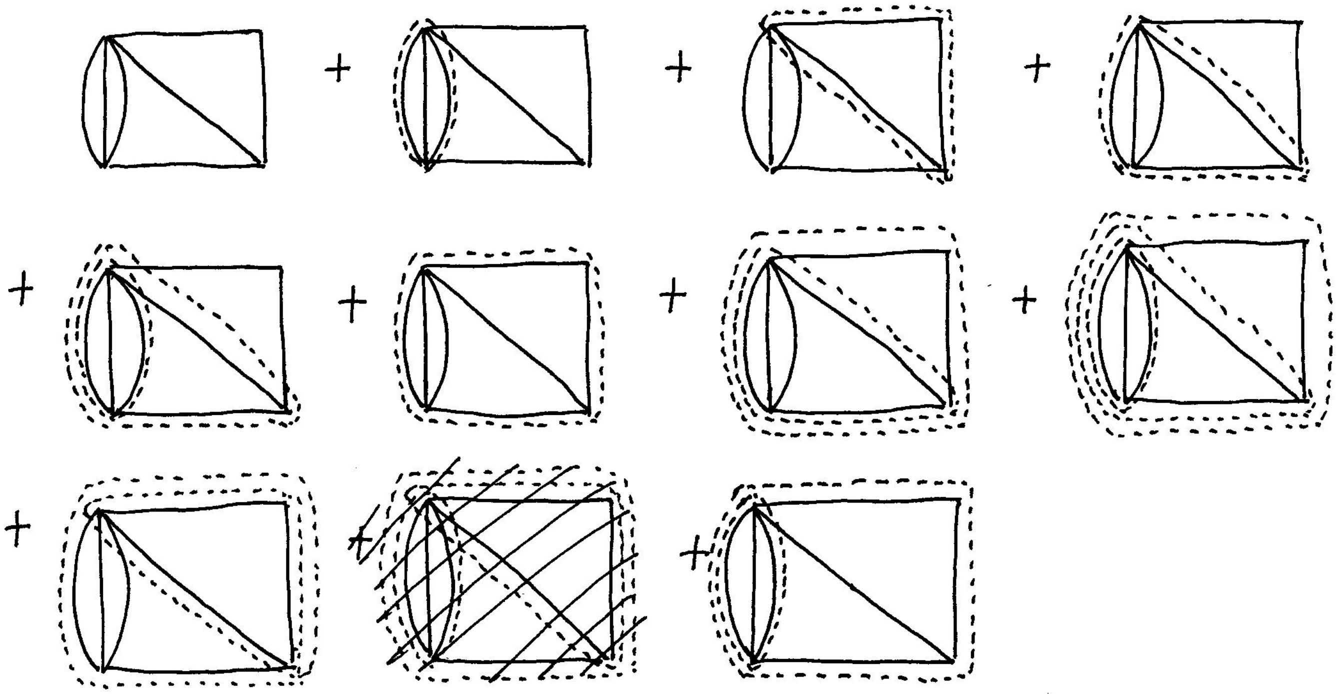
[1] N. N. Bogoliubov, D. V. Shirkov, "Introduction of the Theory of Quantized Fields, Third Edition," (Related chapters: 3, 4, 5, and 6)
[2] K. Hepp, Comm. Math. Phys. **2**, 301 (1966).
[3] W. Zimmerman, Comm. Math. Phys. **15**, 208 (1969).

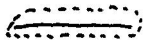
Appendix: An example of R-operation
(From Klaus Hepp's paper, Fig. 2)





$\Rightarrow R(G) =$



$d(G_i)$ denotes coefficient function corresponding to diagram G_i of the counter-term;
dotted line represents $(-M)$-operation;
only non-zero M-operations are kept, e.g. is 1PR and so  $=0$