University of California at San Diego – Department of Physics – Prof. John McGreevy

Whence QFT? (239a) Spring 2014
Assignment 2 – Solutions

Posted April 9, 2014 **Due Wed, April 23, 2014**

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. **Trotterization practice.**

   Here's an exercise in understanding the quantum-to-classical correspondence.

   Suppose we add a term
   $$\Delta\mathbf{H}_1 = -v_x \sum_j \mathbf{X}_j \mathbf{X}_{j+1}$$
   to the hamiltonian of the transverse-field Ising model.

   (a) Show that this term preserves the $\mathbb{Z}_2$ symmetry generated by $\mathbf{S} = \prod_j \mathbf{X}_j$.
   $[\Delta\mathbf{H}_1, \mathbf{S}] = 0$ since both are made from $\mathbf{X}$s.

   (b) Construct a corresponding statistical mechanics model. That is, find C and S so that
   $$\sum_C e^{-S} = \mathrm{tr}_{\mathcal{H}} e^{-\frac{1}{T}(\mathbf{H}_{\mathrm{TFIM}} + \Delta\mathbf{H}_1)}$$
   Are the Boltzmann weights e^{-S} positive?

   The configuration space C is the same as for the TFIM, namely binary variables on the sites of a 2d square lattice. The Boltzmann weight acquires an extra factor of
   $$\prod_{l,i} \langle s_{l+1,i} s_{l+1,i+1} | e^{v_x \frac{\Delta\tau}{T} \mathbf{X}_i \mathbf{X}_{i+1}} | s_{l,i} s_{l,i+1} \rangle \equiv e^{-\sum_{l,i} \Delta S_{l,i}}$$
   The matrix element is
   $$e^{-\Delta S_{l,i}} = \begin{pmatrix} \cosh\tau/T & 0 & 0 & \sinh\tau/T \\ 0 & \cosh\tau/T & \sinh\tau/T & 0 \\ 0 & \sinh\tau/T & \cosh\tau/T & 0 \\ \sinh\tau/T & 0 & 0 & \cosh\tau/T \end{pmatrix}$$
   The fact that there are zeros means that some configurations of the spin system are now *forbidden*, *i.e.* the action S is infinite on these configurations. But the weights are positive.

(c) Answer the previous two questions for

$$\Delta \mathbf{H}_2 = -v_y \sum_j \mathbf{Y}_j \mathbf{Y}_{j+1} \ .$$

Since $\mathbf{XY} = -\mathbf{YX}$,

$$\mathbf{S}\mathbf{Y}_j \mathbf{Y}_{j+1} = (-1)^2 \mathbf{Y}_j \mathbf{Y}_{j+1} \mathbf{S}$$

and so $\mathbf{S}$ is preserved, still.

Using $e^{V \mathbf{Y}_j \mathbf{Y}_{j+1}} = \cosh V + \mathbf{Y}_j \mathbf{Y}_{j+1} \sinh V$, (here $V \equiv v \Delta \tau$) we have

$$\langle s'_j s'_{j+1} | e^{V \mathbf{Y}_j \mathbf{Y}_{j+1}} | s_j s_{j+1} \rangle = \delta_{s,s'} \cosh V - \epsilon_{s_j s'_j} \epsilon_{s_{j+1} s'_{j+1}} \sinh V.$$

Here $\epsilon_{s_1,s_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{s_1,s_2}$. I haven't found a simple expression for the logarithm of this (the action of the stat mech model), but certainly these are not positive Boltzmann weights.

(d) Answer the previous two questions for

$$\Delta \mathbf{H}_3 = -g_y \sum_j \mathbf{Y}_j \ .$$

This breaks $\mathbf{S}$. Using $e^{V \mathbf{Y}_j} = \cosh V + \mathbf{Y}_j \sinh V$, the Boltzmann weights now acquire factors of $\mathbf{i}$.