University of California at San Diego – Department of Physics – Prof. John McGreevy

Whence QFT? (239a) Spring 2014
Assignment 3 – Solutions

Posted April 21, 2014 **Due Mon, May 5, 2014**

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. **Spectral representation.**

   Complete the steps of the derivation of the spectral representation of the finite temperature spin-spin correlation functions described in lecture.

2. **Jordan-Wigner.**

   Solve the following spin chains using the mapping to Majorana fermions.

   In all these problems we will use the following Jordan-Wigner transformation:

$$\chi_1(j) = \mathbf{Z}_j \prod_{k>j} \mathbf{X}_k, \quad \chi_2(j) = \mathbf{Y}_j \prod_{k>j} \mathbf{X}_k \quad .$$

$$\mathbf{X}_j = \mathbf{i}\chi_1(j)\chi_2(j), \qquad \mathbf{Z}_j\mathbf{Z}_{j+1} = -\mathbf{i}\chi_1(j)\chi_2(j+1) \tag{1}$$

   So the majoranas satisfy $\{\chi_\alpha(j), \chi_\beta(l)\} = 2\delta_{\alpha\beta}\delta_{jl}, \alpha, \beta = 1, 2$.

   For comparison: With these conventions the transverse-field ising model hamiltonian is

$$\mathbf{H}_{TFIM} = -J\sum_j (g_x\mathbf{X}_j + g_z\mathbf{Z}_j\mathbf{Z}_{j+1}) = -\mathbf{i}J\sum_j (g_x\chi_1(j)\chi_2(j) - g_z\chi_1(j)\chi_2(j+1))$$

   so that the heisenberg eom are

$$\partial_t\chi_2(j) = -J(g_x\chi_1(j) - g_z\chi_1(j-1)), \partial_t\chi_1(j) = J(g_x\chi_2(j) - g_z\chi_2(j+1)),$$

   and from this we see that in the continuum $\chi_\pm \equiv \frac{1}{2\sqrt{a}}(\chi_1 \pm \chi_2)$ are chiral majorana fermions:

$$(\partial_0 \mp \partial_x)\chi_\pm = m\chi_\mp,$$

   with $m \propto g_z - g_x$.

(a) **XY-model**

$$\mathbf{H} = -J\sum_j (\mathbf{Z}_j\mathbf{Z}_{j+1} + \mathbf{Y}_j\mathbf{Y}_{j+1})$$

I find

$$\mathbf{H} = -J\sum_j (\mathbf{i}\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1) - \mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j+1))$$

$$\mathbf{H} = -J\sum_j \mathbf{i}\,(\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1) + \boldsymbol{\chi}_2(j+1)\boldsymbol{\chi}_1(j))$$

In fourier space,

$$\boldsymbol{\chi}_\alpha(j) = \frac{1}{\sqrt{N}}e^{-\mathbf{i}kaj}\boldsymbol{\chi}_\alpha(k)$$

we get

$$\mathbf{H} = +\mathbf{i}J\sum_k \boldsymbol{\chi}_1(k)\boldsymbol{\chi}_2(-k)2\cos ka = +\sum_k (2J\cos ka)\mathbf{c}_k^\dagger\mathbf{c}_k$$

where $\mathbf{c}_k \equiv \frac{1}{2}(\boldsymbol{\chi}_1(k) + \mathbf{i}\chi_2(-k))$. (Beware my factors of two here.)

This model has a U(1) symmetry which rotates $\mathbf{Z}$ into $\mathbf{Y}$. How does it act on the fermions?

The $\mathsf{U}(1)$ acts by

$$\mathbf{c} \to e^{\mathbf{i}\theta}\mathbf{c}.$$

(b) **Solve an interacting fermion system**

$$\mathbf{H}_{\text{int}} = -J\sum_j (\mathbf{X}_j\mathbf{X}_{j+1} + \mathbf{Y}_j\mathbf{Y}_{j+1}) \tag{2}$$

This model is in fact related by a basis rotation ($\mathbf{U} = \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Y}_j}$) to the one in part 2a.

But if you directly use the mapping we introduced in class in these variables, you'll find quartic terms in the fermions.

The basis transformation above therefore solves this interacting fermion system.

$$\mathbf{H}_{\text{int}} = -J\sum_j (\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j)\mathbf{i}\boldsymbol{\chi}_1(j+1)\boldsymbol{\chi}_2(j+1) + \mathbf{i}\boldsymbol{\chi}_2(j)\boldsymbol{\chi}_1(j+1))$$

In terms of the complex fermions, $\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j) = 1 - 2\mathbf{c}_j^\dagger\mathbf{c}_j = 1 - 2\mathbf{n}_j$ this first term is a near-neighbor density-density interaction, $\propto \mathbf{n}_j\mathbf{n}_{j+1}$.

How does the U(1) symmetry of (2) act on these fermion variables?

It mixes particles and holes.

(c) **A spin chain with a non-onsite Ising symmetry**

Consider the Hamiltonian

$$\mathbf{H} = -J\sum_j (\mathbf{X}_j + \lambda\mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1})$$

i. (Slightly more optional:) Show that when $\lambda = -1$ this model is invariant under the action of

$$\mathbf{S}_1 \equiv \prod_j \mathbf{X}_j \prod_j e^{\mathbf{i}\frac{\pi}{4}\mathbf{Z}_j\mathbf{Z}_{j+1}}.$$

This symmetry is "not-onsite" in that its action on the spin at site j depends on the state of the neighboring sites.

ii. Solve this model by Jordan-Wigner. Show that the spectrum is gapless and that each momentum state is doubly-degenerate.

The chain falls apart into two decoupled pieces, since the ZXZ term only couples odd sites to odd sites and even sites to even sites. Hence the doubling of the spectrum.

iii. [Challenge problem] The previous part shows that this model produces *two* massless majorana fermions of each chirality. Find the action of the $\mathbb{Z}_2$ symmetry on these fermions.

iv. [Challenge problem] Consider the effect of adding the ferromagnetic term $\sum_j \mathbf{Z}_j\mathbf{Z}_{j+1}$ on this system. Is it invariant under the symmetry?

Majorana fermion solution of edge Hamiltonian with non-onsite $\mathbb{Z}_2$ symmetry.

In this problem we consider adding an extra term:

$$\mathbf{H}_2 = -J\sum_j \left(g_x\mathbf{X}_j + g_z\mathbf{Z}_j\mathbf{Z}_{j+1} + \tilde{g}_x\mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1}\right) .$$

When $\tilde{g}_x = -g_x$, this hamiltonian has the symmetry

$$\mathbf{S}_1 = \left(\prod_j \mathbf{X}_j\right)\left(\prod_l e^{\mathbf{i}Q_{l,l+1}}\right)$$

where $e^{\mathbf{i}Q_{l,l+1}} = \sqrt{\mathbf{Z}_l\mathbf{Z}_{l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1-\mathbf{Z}_l\mathbf{Z}_{l+1})}$.

I claim that

$$\mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1} = -\mathbf{i}\chi_1(j-1)\chi_2(j+1) .$$

So this hamiltonian is

$$\mathbf{H}_2 = -\mathbf{i}J\sum_j \left(g_x\chi_1(j)\chi_2(j) - \left(g_z\chi_1(j) - \tilde{g}_x\chi_1(j-1)\right)\chi_2(j+1)\right) .$$

This has the same continuum eom as the usual TFIM model with the replacements: $g_z \to g_z + \tilde{g}_x$ and double the velocity. So the critical point is now at $0 = g_z - \tilde{g}_x - g_x$.

Let's understand the symmetry action on the fermions.

The trivial symmetry action is

$$\mathbf{S}_0 = \prod_j \mathbf{X}_j = i^N \prod_j \chi_1(j)\chi_2(j).$$

This acts as

$$\mathbf{S}_0 \chi_\alpha \mathbf{S}_0^\dagger = -\chi_\alpha,$$

which is indeed a symmetry of $\mathbf{H}_{\text{TFIM}}$.
Using (1), the additional factors in $\mathbf{S}_1$ can be written as:

$$e^{\mathbf{i}Q_{l,l+1}} = e^{\frac{\mathbf{i}\pi}{4}(1+\mathbf{i}\chi_1(j)\chi_2(j+1))} = e^{\frac{\mathbf{i}\pi}{4}} e^{\mathbf{i}\frac{\pi}{4}\mathbf{i}\chi_1(j)\chi_2(j+1)}$$
$$= e^{\frac{\mathbf{i}\pi}{4}} \left(\cos\frac{\pi}{4} + \mathbf{i}\sin\frac{\pi}{4} \mathbf{i}\chi_1(j)\chi_2(j+1) \right) = e^{\frac{\mathbf{i}\pi}{4}} \frac{1 - \chi_1(j)\chi_2(j+1)}{\sqrt{2}}.$$

Notice that this object is indeed unitary:

$$\frac{1-ab}{\sqrt{2}} \left(\frac{1-ab}{\sqrt{2}} \right)^\dagger = \frac{1-ab}{\sqrt{2}} \frac{1+ab}{\sqrt{2}} = \frac{1}{2}(1 - ab + ab - abab) = 1 \ .$$

Acting this on $\chi_\alpha(j)$ gives

$$\mathbf{S}_1 \chi_1(j) \mathbf{S}_1^\dagger \stackrel{?}{=} -\chi_2(j+1) \ . \tag{3}$$

$$\mathbf{S}_1 \chi_2(j) \mathbf{S}_1^\dagger \stackrel{?}{=} \chi_1(j-1) \ .$$

The key step (the only factor in $\prod_j e^{\mathbf{i}Q}$ that matters) comes from

$$\frac{1 \pm \chi_1(j)\chi_2(j+1)}{\sqrt{2}} \chi_2(j+1) \frac{1 \mp \chi_1(j)\chi_2(j+1)}{\sqrt{2}} = \pm\chi_1(j).$$

The essential fact here is the identity:

$$\frac{1 \pm ab}{\sqrt{2}} b \frac{1 \mp ab}{\sqrt{2}} = \pm a, \quad \frac{1 \pm ab}{\sqrt{2}} a \frac{1 \mp ab}{\sqrt{2}} = \mp b,$$

for any two distinct majorana modes a, b. So $\mathbf{S}_1$ seems to act like

$$\begin{pmatrix} 0 & -T \\ T^\dagger & 0 \end{pmatrix}$$

where T is the shift-by-1 operator and the matrix is acting on the α, β space. Notice that this does not seem to square to $\mathbb{1}$! Rather it squares to the operation which reverses the sign of all the majoranas. In the spin chain, this is a gauge symmetry: the sign of the fermion operators is not observable. So there is no contradiction.
Since

$$\mathbf{Z}_{j-1}\mathbf{X}_j\mathbf{Z}_{j+1} = -\mathbf{i}\boldsymbol{\chi}_1(j-1)\boldsymbol{\chi}_2(j+1) \stackrel{(3)}{\mapsto} -\mathbf{i}\left(-\boldsymbol{\chi}_2(j)\right)\left(\boldsymbol{\chi}_1(j)\right) = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j) = -\mathbf{X}_j$$

this $\mathbb{Z}_2$ action indeed preserves the form of $\mathbf{H}_2$ if $g_x = -\tilde{g}_x$ (set $g_z = 0$ for a moment):

$$\mathbf{H}_2^\star = -\mathbf{i}J \sum_j \left(\chi_1(j)\chi_2(j) - \chi_1(j-1)\chi_2(j+1) \right) \text{ has } 0 = [\mathbf{S}_1, \mathbf{H}_2^\star] \ .$$

In the continuum limit, if we ignore the shift, the transformation (3) is basically $\chi_1 \to \chi_2, \chi_2 \to \chi_1$, which acts on $\chi_\pm$

$$\mathbf{S}_1 : \chi_+ \to \chi_+, \quad \chi_- \to -\chi_- \ ,$$

which indeed acts nontrivially only on χ_-. This is a chiral symmetry.

More microscopically, it seems that we should define the chiral majoranas to be

$$\chi_\pm(j) \sim \chi_1(j) \pm \chi_2(j+1).$$

Notice that this regrouping is very much like the dual jordan-wigner fermions. That is: if I relabel my degrees of freedom as living on the links as follows:

$$\gamma_1(j+\frac{1}{2}) \equiv \chi_1(j), \gamma_2(j+\frac{1}{2}) = -\chi_2(j+1)$$

then in terms of the gammas, the TFIM hamiltonian has the roles of the X and Z terms reversed:

$$\mathbf{H}_{TFIM} = -\mathbf{i}J\sum_j \left(g_x\gamma_1(j)\gamma_2(j+1) - g_z\gamma_1(j)\gamma_2(j)\right)$$

which if I then rewrite in terms of new spin variables amounts to a duality transformation, i.e. produces the original hamiltonian with the replaxement $g_x/g_z \to g_z/g_x$.

Addendum: The preceding discussion is correct, *except*: a 1d chain with *only* next-nearest-neighbor hopping falls apart into two decoupled chains: odd sites only couple to odd sites and even sites only couple to even sites. This means we actually get two copies of the majorana with this chiral realization of the symmetry.

In fact, the ferromagnetic term is also invariant under $\mathbf{S}_1$. This is clear in terms of the spins, since the extra phase factors commute with $\mathbf{Z}$s. And in fact the extra phase factors are *made* from the combination $\mathbf{Z}_j\mathbf{Z}_{j+1}$. In terms of the majoranas,

$$\mathbf{Z}_j\mathbf{Z}_{j+1} = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_2(j+1) \mapsto -\mathbf{i}\left(-\boldsymbol{\chi}_2(j+1)\right)\left(+\boldsymbol{\chi}_1(j)\right) = -\mathbf{i}\boldsymbol{\chi}_1(j)\boldsymbol{\chi}_j(j+1).$$

Adding this term will gap out the majorana fields (*i.e.* it adds a term in the eom which is not proportional to k). *However*: here will still be two degenerate groundstates. With an open chain, these are the dangling majorana modes at the ends. This is because with both $g_z \neq 0$ and $\tilde{g}_x \neq 0$ (both of which couple j to $j+\frac{1}{2}$), we will always be in the regime (the ferromagnetic phase) where the pairing is between site j and site $j+\frac{1}{2}$, leaving out the sites $\frac{1}{2}$ and $N+\frac{1}{2}$.

This model is discussed in this paper by Xie Chen et al.

(d) **Kitaev-honeycomb-model-like chain**

Consider

$$\mathbf{H}_\mathrm{K} = \sum_j \left(\mathbf{X}_{2j}\mathbf{X}_{2j+1} + \mathbf{Y}_{2j}\mathbf{Y}_{2j-1}\right)$$

where the bonds alternate between XX interactions and YY interactions. There are now two sites per unit cell, which means that the solution in terms of momentum-space fermion operators will involve two bands. Find their dispersion.

$$\mathbf{H}_{\mathrm{K}} = \sum_j \mathbf{i}\left(-\boldsymbol{\chi}_1(2j)\boldsymbol{\chi}_2(2j+1) + \boldsymbol{\chi}_2(2j)\boldsymbol{\chi}_1(2j-1)\right)$$

Introduce fourier modes for the bravais lattice with two sites (hence four majoranas) per unit cell

$$\begin{pmatrix} \boldsymbol{\chi}_1(2j) \\ \boldsymbol{\chi}_2(2j+1) \\ \boldsymbol{\chi}_1(2j+1) \\ \boldsymbol{\chi}_2(2j) \end{pmatrix} = \frac{1}{\sqrt{N}} e^{\mathbf{i}k(2ja)} \boldsymbol{\chi}_A(k)$$

whose hamiltonian is

$$\mathbf{H}_{\mathrm{K}} = \sum_k \mathbf{i}\boldsymbol{\chi}_A(k)\boldsymbol{\chi}_B(-k) t_{AB}(k)$$

with

$$t_{AB}(k) = \begin{pmatrix} 0 & e^{\mathbf{i}ka} & 0 & 0 \\ e^{-\mathbf{i}ka} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Since the bottom block of this matrix doesn't depend on k, one of the bands is flat.