

Whence QFT? (239a) Spring 2014 Assignment 3

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Due Mon, May 5, 2014

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. Spectral representation.

Complete the steps of the derivation of the spectral representation of the finite temperature spin-spin correlation functions described in lecture.

2. Jordan-Wigner.

Solve the following spin chains using the mapping to Majorana fermions.

(a) XY-model

$$\mathbf{H} = -J \sum_j (\mathbf{Z}_j \mathbf{Z}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1})$$

This model has a U(1) symmetry which rotates \mathbf{Z} into \mathbf{Y} . How does it act on the fermions?

(b) Solve an interacting fermion system

$$\mathbf{H} = -J \sum_j (\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1}) \quad (1)$$

This model is in fact related by a basis rotation ($\mathbf{U} = \prod_j e^{i\frac{\pi}{4}\mathbf{Y}_j}$) to the one in part 2a.

But if you directly use the mapping we introduced in class in these variables, you'll find quartic terms in the fermions.

The basis transformation above therefore solves this interacting fermion system. How does the U(1) symmetry of (1) act on these fermion variables?

(c) A spin chain with a non-onsite Ising symmetry

Consider the Hamiltonian

$$\mathbf{H} = -J \sum_j (\mathbf{X}_j + \lambda \mathbf{Z}_{j-1} \mathbf{X}_j \mathbf{Z}_{j+1})$$

- i. (Slightly more optional:) Show that when $\lambda = -1$ this model is invariant under the action of

$$\mathbf{S}_1 \equiv \prod_j \mathbf{X}_j \prod_j e^{i\frac{\pi}{4} \mathbf{Z}_j \mathbf{Z}_{j+1}}.$$

This symmetry is “not-onsite” in that its action on the spin at site j depends on the state of the neighboring sites.

- ii. Solve this model by Jordan-Wigner. Show that the spectrum is gapless and that each momentum state is doubly-degenerate.
- iii. [Challenge problem] The previous part shows that this model produces *two* massless majorana fermions of each chirality. Find the action of the \mathbb{Z}_2 symmetry on these fermions.
- iv. [Challenge problem] Consider the effect of adding the ferromagnetic term $\sum_j \mathbf{Z}_j \mathbf{Z}_{j+1}$ on this system. Is it invariant under the symmetry?
- (d) **Kitaev-honeycomb-model-like chain**

Consider

$$\mathbf{H} = \sum_j (\mathbf{X}_{2j} \mathbf{X}_{2j+1} + \mathbf{Y}_{2j} \mathbf{Y}_{2j-1})$$

where the bonds alternate between XX interactions and YY interactions. There are now two sites per unit cell, which means that the solution in terms of momentum-space fermion operators will involve two bands. Find their dispersion.