University of California at San Diego – Department of Physics – Prof. John McGreevy

Whence QFT? (239a) Spring 2014
Assignment 4 – Solutions

Posted May, 3 2014 **Due Wed, May 14, 2014**

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. **Homage to Onsager.**

   Show that the groundstate energy of Ising chan with $N \gg 1$ sites may be written as

   $$E_0(g) = -NJ \int_0^\pi \frac{dk}{2\pi} \epsilon_k$$

   where ϵ_k is the dispersion we derived for the fermions.

   Show that this can be written as

   $$\frac{1}{NJ} E_0(g) = -\frac{2}{\pi}(1+g)E(\pi/2, \sqrt{1-\gamma^2}), \quad \gamma = \left| \frac{1-g}{1+g} \right|$$

   (notice that this expression is manifestly self-dual) where $E(\pi/2, x)$ is the elliptic integral

   $$E(\pi/2, x) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - x^2 \sin^2 \theta}.$$

   Expand this result in $g - g_c$.

   This calculation is done pretty explicitly in Fradkin's 2d edition, page 125.

2. **Grassmann problems.**

   (a) A useful device is the integral representation of the grassmann delta function. Show that

   $$-\int d\bar{\psi}_1 e^{-\bar{\psi}_1(\psi_1 - \psi_2)} = \delta(\psi_1 - \psi_2)$$

   in the sense that $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(\psi_2)$ for any grassmann function f.

   (b) Recall the resolution of the identity on a single qbit in terms of fermion coherent states

   $$\mathbb{1} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} |\psi\rangle\langle\bar{\psi}|. \tag{1}$$

   Show that $\mathbb{1}^2 = \mathbb{1}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\text{tr}\mathbf{A} = \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \langle -\bar{\psi}|\mathbf{A}|\psi\rangle \; ,$$

and the minus sign in the bra had important consequences.
(Here $\langle -\bar{\psi}|\mathbf{c}^\dagger = \langle -\bar{\psi}|(-\bar{\psi})$).
Check that using this expression you get the correct answer for

$$\text{tr}(a + b\mathbf{c}^\dagger \mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (1) by expanding the coherent states in the number basis.
Using $|\psi\rangle = |0\rangle + \psi|1\rangle, \langle -\bar{\psi}| = \langle 0| - \bar{\psi}\langle 1|$, we have

$$\begin{aligned}
\int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} |\psi\rangle\langle\bar{\psi}| &= \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left(|0\rangle + \psi|1\rangle\right) \left(\langle 0| - \bar{\psi}\langle 1|\right) \\
&= \int d\bar{\psi} d\psi \; e^{-\bar{\psi}\psi} \left(|0\rangle\langle 0| - \psi\bar{\psi}|1\rangle\langle 1|\right) \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}. \qquad (2)
\end{aligned}$$

3. **Heisenberg chain**

Consider the Heisenberg hamiltonian

$$\mathbf{H} = -J \sum_j \left(\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} + v \mathbf{Z}_j \mathbf{Z}_{j+1}\right) \; .$$

When $v = 1$ there is $\mathsf{SU}(2)$ symmetry. What are the generators?

The generators are just the Pauli operators $\sum_j \vec{\boldsymbol{\sigma}}_j$.

On the previous problem set we successfully fermionized the model with $v = 0$. Fermionize the v term.

We saw (in a different basis) that this is a 4-fermion term. It is

$$\left(2\mathbf{c}_j^\dagger \mathbf{c}_j - 1\right) \left(2\mathbf{c}_{j+1}^\dagger \mathbf{c}_{j+1} - 1\right) .$$

Take the continuum limit.

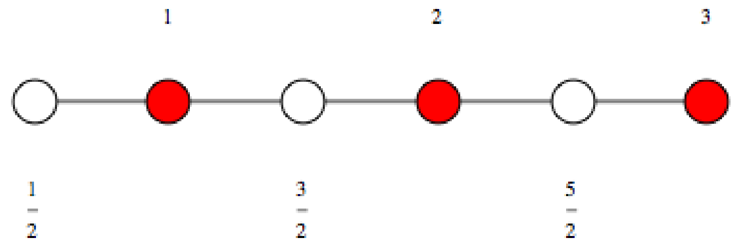
Plugging in $\Psi(x_j) = \frac{1}{\sqrt{a}} \mathbf{c}_j$ gives

$$\begin{aligned}
\Delta\mathbf{H} &\simeq -J \int dx \left(2\Psi^\dagger(x)\Psi(x) - 1\right) \left(2\Psi^\dagger(x+a)\Psi(x+a) - 1\right) \\
&= -Ja^2 \int dx \left(2\Psi^\dagger(x)\Psi(x)\right) \left(2\partial\Psi^\dagger(x)\partial\Psi(x)\right) + \text{quadratic terms} \qquad (3)
\end{aligned}$$

Fermi statistics get rid of the term with fewer derivatives.

4. **Counting of states**

Consider the gauge theory description of the transverse field Ising model with $N = 3$ sites.



With open boundary conditions, find an explicit expression for the gauge transformation $\{s_{\bar{j}}\}$ which eliminates the original $\mathbf{X}_j$ variables (sets them to 1).

Since $\mathbf{X}_{\bar{j}} \to s_{\bar{j}-\frac{1}{2}} \mathbf{X}_{\bar{j}} s_{\bar{j}+\frac{1}{2}}$, we require

$$1 = s_{\frac{1}{2}} \mathbf{X}_1 s_{\frac{3}{2}}, \; 1 = s_{\frac{3}{2}} \mathbf{X}_2 s_{\frac{5}{2}}, \; 1 = s_{\frac{5}{2}} \mathbf{X}_3$$

– there is no site to the right of $\mathbf{X}_3$. So set $s_{5/2} = \mathbf{X}_3$, $s_{3/2} = \mathbf{X}_2 \mathbf{X}_3$, $s_{\frac{1}{2}} = \mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3$ and we're done.

With closed boundary conditions, show that you cannot eliminate $\prod_j \mathbf{X}_j$. Find an expression for the gauge choice which eliminates all but one of the spins.

Nothing we do with our silly ss can eliminate $\prod_j \mathbf{X}_j$ because it is gauge invariant.