

Whence QFT? (239a) Spring 2014 Assignment 4

Posted May, 3 2014

Due Wed, May 14, 2014

All problems are optional in the following sense: if you are sure that you know the ideas involved so well that it would be a waste of your time to do the problem, don't do it, or merely sketch the answer. By this point in your education you don't need to rely on me to determine what you know and don't know.

1. Homage to Onsager.

Show that the groundstate energy of Ising chain with $N \gg 1$ sites may be written as

$$E_0(g) = -NJ \int_0^\pi \frac{dk}{2\pi} \epsilon_k$$

where ϵ_k is the dispersion we derived for the fermions.

Show that this can be written as

$$\frac{1}{NJ} E_0(g) = -\frac{2}{\pi} (1+g) E(\pi/2, \sqrt{1-\gamma^2}), \quad \gamma = \left| \frac{1-g}{1+g} \right|$$

(notice that this expression is manifestly self-dual) where $E(\pi/2, x)$ is the elliptic integral

$$E(\pi/2, x) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - x^2 \sin^2 \theta}.$$

Expand this result in $g - g_c$.

2. Grassmann problems.

- (a) A useful device is the integral representation of the grassmann delta function. Show that

$$- \int d\bar{\psi}_1 e^{-\bar{\psi}_1(\psi_1 - \psi_2)} = \delta(\psi_1 - \psi_2)$$

in the sense that $\int d\psi_1 f(\psi_1) \delta(\psi_1 - \psi_2) = f(\psi_2)$ for any grassmann function f .

- (b) Recall the resolution of the identity on a single qbit in terms of **fermion coherent states**

$$\mathbb{1} = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\bar{\psi}|. \tag{1}$$

Show that $\mathbb{1}^2 = \mathbb{1}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was

$$\text{tr} \mathbf{A} = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle -\bar{\psi} | \mathbf{A} | \psi \rangle ,$$

and the minus sign in the bra had important consequences.

(Here $\langle -\bar{\psi} | \mathbf{c}^\dagger = \langle -\bar{\psi} | (-\bar{\psi})$).

Check that using this expression you get the correct answer for

$$\text{tr}(a + b\mathbf{c}^\dagger\mathbf{c})$$

where a, b are ordinary numbers.

(d) Prove the identity (1) by expanding the coherent states in the number basis.

3. Heisenberg chain

Consider the Heisenberg hamiltonian

$$\mathbf{H} = -J \sum_j (\mathbf{X}_j \mathbf{X}_{j+1} + \mathbf{Y}_j \mathbf{Y}_{j+1} + v \mathbf{Z}_j \mathbf{Z}_{j+1}) .$$

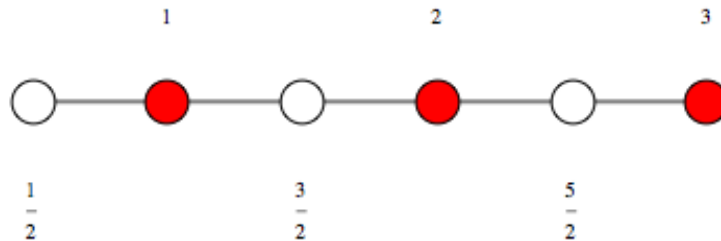
When $v = 1$ there is $\text{SU}(2)$ symmetry. What are the generators?

On the previous problem set we successfully fermionized the model with $v = 0$. Fermionize the v term.

Take the continuum limit.

4. Counting of states

Consider the gauge theory description of the transverse field Ising model with $N = 3$ sites.



With open boundary conditions, find an explicit expression for the gauge transformation $\{s_{\bar{j}}\}$ which eliminates the original \mathbf{X}_j variables (sets them to 1).

With closed boundary conditions, show that you cannot eliminate $\prod_j \mathbf{X}_j$. Find an expression for the gauge choice which eliminates all but one of the spins.