Braiding Statistics in Three Dimensions

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In this paper, I summarize the results obtained by a recent study of the braiding statistics of loop excitations in three dimensions. This is seen to be an extension of the anyonic braiding statistics in two dimensions. Thus, fermionic and bosonic excitations are just a specific case of this new statistics.

INTRODUCTION

In $\mathbb{R}^3$, the spin-statistics theorem classifies particles as either fermions or bosons, depending on whether the wavefunction under the exchange of two such particles is anti-symmetric or symmetric, respectively. In $\mathbb{R}^2$, particle exchange picks up a continuum of values, rather than the discrete ±1 that we see in $\mathbb{R}^3$. These particles are called *anyons*, so called because they can take on any phase under particle exchange, i.e.:

$$|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle, \text{ for particles in } \mathbb{R}^2 \tag{1}$$

For particles in $\mathbb{R}^3$, $\theta \in \{0, \pi\}$. It is then natural to ask if we can generalize the braiding statistics in $\mathbb{R}^3$ so that objects picks up a general phase when interchanged (i.e. $\theta \in [0, 2\pi)$).

The authors of [1] consider the braiding statistics between particle-like excitations and loop-like excitations which live on a 3D lattice. At first thought, particle-loop and two-loop braids may appear to be sufficient to encapsulate the braiding statistics. The authors of [1] show that this is not true: that in loop-loop braids, a third loop is necessary in order to obtain physically relevant, non-trivial statistical phases. See Figure 1.

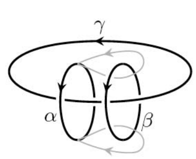

FIG. 1: Loop α is braided around loop β, both of which have loop γ going through them. Image from [1].

The authors of [1] consider a 3D lattice built out of K different species of bosons whose particle numbers are conserved (mod N). In other words, there are K independent local operators which, when acting on states, return values that are modular. As such, this is a $(\mathbb{Z}_N)^K$ gauge theory.

Each edge on the lattice corresponds to particle spin, so there are various unitary operators for this system:

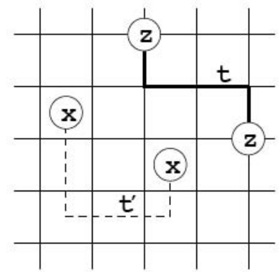

FIG. 2: Electric excitations terminate on vertices; magnetic excitations terminate in plaquettes. Image from [2].

$$\begin{aligned} \hat{S}_{\pm}|n\rangle &= e^{\pm 2\pi i n/N}|n\rangle \\ \hat{W}_{\pm}|n\rangle &= |n \pm 1 \ (modN)\rangle \end{aligned} \tag{2}$$

where $n = 0, 1, \ldots, N-1$, and where $\hat{S}$ measures spin and $\hat{W}$ raises or lowers spin (see [3] for additional information on the algebra of these operators).

States in the Hilbert space can be thought of as strings on the lattice. Point-like excitations carry gauge charge and occur for strings which do not close. "Electric" excitations occur at the ends of strings which terminate on nodes of the lattice; "magnetic" excitations occur at the ends of strings which terminate in the plaquettes of the lattice (see Figure 2). The most general point-like excitation in this theory is then $q = (q_1, \ldots, q_K)$, where each q_i is conserved (mod N). There can also be loop-like excitations which carry gauge flux $\phi = (\phi_1, \ldots, \phi_K)$, where each ϕ_i is some multiple of $\frac{2\pi}{N}$ [5].

BRAIDING STATISTICS

We can now proceed to answer the question of what the resulting statistical phase is when we use charges and loops in this theory for the braiding process. There are three braiding processes we can consider: charge-charge, charge-loop, or loop-loop. Since the particle excitations are bosons, the charge-charge process gives no statistical phase.

Charge-loop braiding Statistics

If a charge q is braided around a loop with flux ϕ, the resulting statistical phase (θ) is given by the Aharanov-Bohm equation:

$$\theta = q \cdot \phi \tag{3}$$

Two-loop braiding statistics

See Figure 3. There are two interesting cases here that will give different statistical phases: (1) loops α and β are neutral or (2) α and β are charged. For case (1), either loop can be shrunk to a point and annihilated, so that the other loop would simply braid around vacuum. This gives $\theta = 0$. For case (2), using equation (3) we get $\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha$, because loop β "sees" charge q_α braid around it and vice versa for loop α.

Although this phase is non-trivial, it is independent of the actual bosons we choose to populate the lattice. Thus, we should move on to examine if we can construct statistical phases which differ depending on the SPT model from the three-loop braiding process.

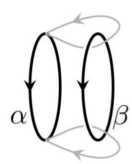

FIG. 3: Loop α is braided around loop β. Image from [1].

Three-loop braiding statistics

Adding a third loop that passes through α and β alters the topology of the braiding process, so we may expect to see a different statistical phase arise (see Figure 1). There are two statistical phases to consider here: $\theta_{\alpha\beta,\gamma}$ and $\theta_{\alpha,\gamma}$. The first corresponds to "full-braiding" two distinct loops (i.e. $\alpha \neq \beta$), the second corresponds to "half-braiding" two identical loops. The braid operation is taken to be commutative so that $\theta_{\alpha\beta,\gamma} = \theta_{\beta\alpha,\gamma}$. Another constraint is that $\theta_{\alpha\alpha,\gamma} = \theta_{\alpha,\gamma} + \theta_{\alpha,\gamma} = 2\theta_{\alpha,\gamma}$ (i.e. two half-braids = one full-braid).

Further constraints on θ can be imposed if additional algebraic operators are defined on loops: $\beta_1 + \beta_2$ and $\beta_1 \oplus \beta_2$ (Figure 4). The first operation fuses two loops together that initially share the same γ-loop, and results in a final loop with the same γ-loop going through it. The second operation fuses two loops together that initially have different γ-loops, and results in a final loop with both initial γ-loops going through it. These lead to the further constraints (proved in [1]):

$$\begin{aligned} \theta_{\alpha(\beta_1+\beta_2),\gamma} &= \theta_{\alpha\beta_1,\gamma} + \theta_{\alpha\beta_2,\gamma} \\ \theta_{(\alpha_1 \bigoplus \alpha_2)(\beta_1 \bigoplus \beta_2),(\gamma_1+\gamma_2)} &= \theta_{\alpha_1\beta_1,\gamma 1} + \theta_{\alpha_2\beta_2,\gamma 2} \end{aligned} \tag{4}$$

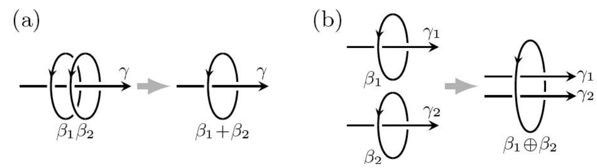

FIG. 4: Two ways to add loops. Image from [1].

For loops α and β with unit flux, we can define: $\Theta_{ij,k} \equiv N\theta_{\alpha\beta,e_k}$ and $\Theta_{i,k} \equiv N\theta_{\alpha,e_k}$, where $\phi_\alpha = \frac{2\pi}{N}e_i$, $\phi_\beta = \frac{2\pi}{N}e_k$, $e_l \equiv (0,\ldots,1,\ldots,0)$=vector with a "1" in the lth coordinate, and zeroes everywhere else. Using these new objects and the constraints in (4), we should be able to distinguish between different symmetry-protected topological models. The authors of [1] derive the following:

$$\begin{aligned} \Theta_{ij,k} &= \frac{2\pi}{N}(M_{ikj} - M_{kij} + M_{jki} - M_{kji}) \\ \Theta_{i,j} &= \frac{2\pi}{N}(M_{iji} - M_{jii}) \end{aligned} \tag{5}$$

Different 3D gauged SPT models can be constructed by elements of the cohomology group $H^4((\mathbb{Z}_N)^K, U(1))$ (see [4]). The M_{ijk} are used to construct an Abelian subgroup of H^4, so they contain information about the specific gauged SPT model. For instance, $(\mathbb{Z}_N)^2$ gives N^2 distinct phases. For a more complete description of these braiding statistics, it may be necessary to look at the non-Abelian elements of H^4.

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[5] Note that charge can be attached to these loops