Symmetry breaking vs topological order: weak symmetry breaking

Alex Kuczala[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

Although we think of topological and symmetry broken order as two distinct kinds of order, this paper describes a situation in which this distinction blurs. We describe a configuration in which we gap the edge of a topological insulator with a nonlocal order parameter that breaks time reversal symmetry. Rather than having twofold symmetry broken degeneracy, the ground state has topologically protected fourfold degeneracy. We show that in the limit of a thin strip, the nonlocal order parameter becomes local, ground state becomes twofold degenerate symmetry broken state.

INTRODUCTION

Topological insulators have gapless edge modes protected by charge conservation and time reversal symmetry. The edge of a topological insulator can be gapped only if either of these symmetries are broken, explicitly or spontaneously. However, Wang and Levin [4] consider an apparently paradoxical situation where it seems that we can gap the edge of a topological insulator without breaking any symmetries. This system, shown in Figure 1c, consists of a topological insulator surrounded by a fractional insulator annulus, which is in turn surrounded by vacuum. The resolution of this paradox introduces a 'weak' kind of symmetry breaking with a nonlocal order parameter. When the width of the fractional insulator becomes small, the order parameter becomes local, and we recover the usual notion of symmetry breaking.

First, we do a lightning fast review of Chern-Simons theory and the K-matrix formalism, which may be skipped [7]. We then examine the two interfaces and see how they can be gapped without breaking any symmetries, leading to an apparent paradox: a topological insulator with a gapped edge with unbroken symmetry. We resolve the paradox by finding a nonlocal order parameter that breaks time reversal symmetry. We then examine the degeneracy of the ground state, which has a topological origin.

CHERN-SIMONS THEORY

We can conveniently describe 2+1 topological insulators and fractional quantum hall fluids using Chern-Simons theory. Chern-Simons theory uses an effective gauge field to describe the mutual statistics of quasiparticles. Since conserved current has no divergence ($\partial_\mu J^\mu = 0$), in 3 dimensions one can write the current as the curl of another vector (gauge) field a [6]. Chern-Simons theory has a topological action that attaches 'magnetic' flux to charges. When these charges move around each other, they pick up an Aharanov-Bohm phase due to the flux. In this way, the quasiparticles acquire nontrivial mutual statistics.

The Chern-Simons action for N species of quasiparticles is

$$S_{CS} = \frac{1}{4\pi}\int d^2xdt\left[\sum_{IJ}\epsilon^{\cdots}a^I_{\cdot}K_{IJ}\partial_{\cdot}a^J_{\cdot} - \sum_I a^I\cdot j_I\right] \tag{1}$$

Here the indices are too small to see. The symmetric $N\times N$ integer K matrix describes N the mutual statistics of N species of quasiparticles with currents J_I. Quasiparticles $\sum_I l_I a^I$ are described by the integer vector l.

We can determine the self statistics

$$\theta = \pi l^T K^{-1} l_2$$

of each quasiparticle, and the mutual statistics between quasiparticles l_1 and l_2

$$\theta_{12} = 2\pi l_1^T K^{-1} l_2 \tag{2}$$

where $e^{i\theta}$ is the phase accrued during the exchange.

We can give the quasiparticles electromagnetic charges by adding a term coupling the internal fields a to the electromagnetic field A:

$$L = \frac{-q}{2\pi}\epsilon^{\cdots}t_I A_{\cdot}\partial_{\cdot}a^I_{\cdot}$$

each quasiparticle l has charge

$$Q = qt^T K^{-1} l$$

and the Hall conductance is

$$\sigma_{xy} = \nu\frac{q^2}{2\pi} = \frac{q^2}{2\pi}t^T K^{-1} t$$

Chern-Simons theory also gives us a description of the boundary. The Chern-Simons term $\epsilon a\partial a$ does not vanish at the boundary, and gives us edge modes with action [4]

$$S_{edge} = \frac{1}{4\pi}\int dtdx\sum_{IJ}\left[K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J\right] \tag{3}$$

The number of positive and negative eigenvalues of K determines the number of left and right movers on the edge. The matrix V depends on microscopic physics rather than the bulk properties and determines the velocities of the edge modes. The operator

$$e^{l^T\Phi} \tag{4}$$

creates a quasiparticle l at the edge, where $\Phi = (\phi_1, \ldots \phi_N)$ [4].

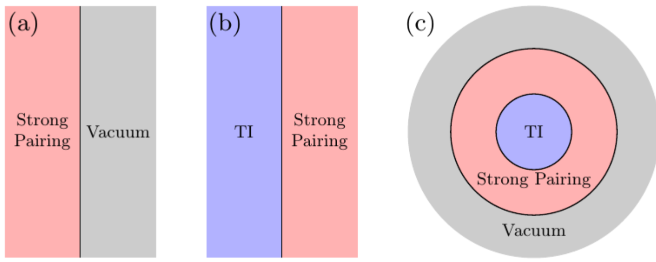

FIG. 1: The interfaces of the system. The topological insulator (TI), strong pairing insulator, and vacuum are shown in blue, pink, and gray, respectively. Taken from [4]

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GAPPING OUT THE INTERFACES

We first consider the boundary between the strongly paired insulator and the vacuum. We construct the strongly paired insulator (SPI) by superimposing two $\nu = \frac{1}{2}$ FQH states, one of each chirality. Start with a free electron gas and separate the spin species with a magnetic field. Then each spin species has opposite chirality. For each spin species, Cooper pair the electrons to form two effective Laughlin states with bosonic excitations $b_\uparrow^\dagger$, $b_\downarrow^\dagger$. The Cooper pairs have effectively half the density of the fermions, and twice the charge, so they have effective filling fraction $\nu = \frac{1}{8}$ [1]. The strongly paired insulator then has K-matrix

$$K_{SP} = \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}$$

with 8 and -8 denoting each chirality. We can describe Cooper pair excitations with quasiparticles $l = (8,0)$ (spin $\uparrow$) and $l = (0,8)$ (spin $\downarrow$). You can check that these have charge $2e$ and bosonic statistics. The Cooper pair edge modes are then (4) $b_\uparrow^\dagger = e^{8i\pi\phi_1}$ and $b_\downarrow^\dagger = e^{-8i\pi\phi_2}$. The Cooper pairs transform under time reversal as

$$b_\uparrow^\dagger \to b_\downarrow^\dagger \qquad b_\downarrow^\dagger \to b_\uparrow^\dagger$$

so we can choose ϕ_1 and ϕ_2 to transform as

$$\phi_1 \to \phi_2 \qquad \phi_2 \to \phi_1$$

under time reversal.

Next, we determine a perturbation to the edge which may gap the system without breaking charge or time reversal symmetry. A spin flip of two Cooper pairs certainly does not break either symmetry:

$$b_\uparrow^\dagger b_\downarrow + b_\uparrow b_\downarrow^\dagger \sim \cos(\Lambda_1^T K\Phi - \alpha(x))$$

where $\Lambda_1^T = (1,-1)$, and α is an arbitrary function.

Now consider the interface b between the topological insulator (TI) and SPI (Figure 1). We model the topological insulator using two $\nu = 1$ electronic quantum hall states with opposite chiralities [2]. Its K-matrix is

$$K_{TI} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We label the edge modes of the TI ϕ_5, ϕ_6. The creation operators for electrons on the edge of the TI are $\psi_\uparrow^\dagger = e^{i\phi_5}$ and $\psi_\downarrow^\dagger = e^{-i\phi_6}$

Under time reversal, the electrons to transform as

$$\psi_\uparrow^\dagger \to \psi_\downarrow^\dagger, \qquad \psi_\downarrow^\dagger \to -\psi_\uparrow^\dagger$$

so we choose

$$\phi_5 \to \phi_6, \qquad \phi_6 \to \phi_5 + \pi$$

The SPI also has edge modes ϕ_3 and ϕ_4 here. We can write the K matrix for all the modes on edge b as

$$K_b = \begin{pmatrix} K_{SPI} & 0 \\ 0 & K_{TI} \end{pmatrix}$$

with $\Phi_b = (\phi_3, \phi_4, \phi_5, \phi_6)$. The interaction $b_\downarrow^\dagger \psi_\uparrow^\dagger (\psi_\downarrow)^3 + h.c.$ conserves charge, but is not time reversal invariant. Under time reversal, the spins in the interaction are flipped and the term acquires a minus sign. Adding the time reversed interaction, the sum of the two terms is both charge conserving and time reversal invariant. We can write this term compactly as

$$\cos(\Lambda_2^T K\Phi - \alpha) - \cos(\Lambda_3^T K\Phi - \alpha)$$

where

$$\Lambda_2^T = (0,1,1,-3) \qquad \Lambda_3^T = (-1,0,3,-1)$$

This interaction therefore preserves charge and time reversal symmetry.

In order for these interactions to create an energy gap, they must lock each of the fields Φ to a minimum of the cosine potential. It turns out this happens only when $\Lambda_i^T K \Lambda_i = 0$ for each term [4]. This holds for the two terms we have considered.

Another thing to consider - although these interaction terms do not explicitly break the symmetry, they could cause spontaneous breaking of the symmetry. The symmetry is spontaneously broken if there is a local order parameter that transforms nontrivially under the symmetry, and acquires a vev. This condition amounts to requiring that the Λ_i are not multiples of any other integer vector (Λ_i is 'primitive') [4]. If Λ was a multiple of another Λ', then although $\cos(\Lambda K \Phi)$ would be invariant under the symmetry, the order parameter $\cos(\Lambda' K\Phi)$ would not in general be invariant. Λ_1, Λ_2 and Λ_3 are primitive, so the symmetry is not spontaneously broken on either edge.

A PARADOX? AND ITS RESOLUTION

Combining these two interfaces on an annulus leads to an apparent contradiction. Consider a TI disk surrounded by a SPI annulus, which forms an interface with the vacuum (Figure 1). This system contains both interfaces above: an inner interface b between the TI and SPI, and the outer interface a between the SPI and vacuum. We showed that both interfaces a and b could be locally gapped while preserving charge and time reversal symmetry. It therefore seems as though we have created a gapped interface (the SPI) between the TI and the vacuum without breaking any symmetries, which contradicts the fact that the inner disk is a TI: it should have a gapless edge! How is it that we can gap out both edges without breaking the symmetry?

Answer: the symmetry is in fact spontaneously broken! A nonlocal order parameter W, responsible for tunneling between the two interfaces, breaks time reversal symmetry. However, the system doesn't have two ground states as one expects from breaking time reversal symmery - it has four. Wang and Levin call this phenomenon weakly broken symmetry.

To show that time reversal symmetry is in fact broken, we explicitly construct W. Consider the operator

$$W \sim \cos(2\phi_1 + 2\phi_2 + 2\phi_3 + 2\phi_4 + \phi_5 + \phi_6)$$

which can be written

$$W \sim \cos(\Lambda_1 K_a \Phi_a + \Lambda_2 K_b \Phi_b + \Lambda_3 K_b \Phi_b)$$

This interaction acquires a minus sign under time reversal. The interactions we considered on each of the edges lock the terms in the cosine above, so $\langle W \rangle \neq 0$. Furthermore, since the SPI bulk is gapped and the operator W acts on both edges, W describes a nonlocal tunneling operator. This operator describes a $l = (2, 2)$ quasiparticle tunneling from boundary b to a, along with an electron spin flip at boundary b. Since W describes a tunneling process, it depends exponentially on the width of the annulus $\sim e^{-D}$. In the limit that the annulus is smaller than the correlation length, W becomes a local order parameter, 'strongly' breaking the symmetry in the usual way.

GROUND STATE DEGENERACY

We would expect a system with spontaneously broken time reversal symmetry to have a twofold degenerate ground state. However, the ground state is fourfold degenerate, which we will see has a topological origin. To investigate, we consider two nonlocal string operators A_l and B_l shown in Figure 2. Operator A creates a quasiparticle l and its antiparticle $\bar{l}$, moves l around the annulus,

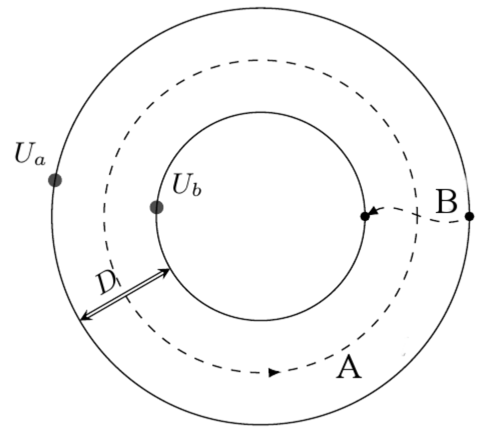

FIG. 2: String operators A and B on the SPI annulus. Adapted from [4].

and then annihilates the two particles. Operator B creates a quasiparticle/hole pair, moves l to the inner edge and $\bar{l}$ to the outer edge, and locally annihilates them. A and B are unaffected by the paths the quasiparticles take. These two operators don't commute - they satisfy [4]

$$A_l B_{l'} = e^{\theta_{ll'}} B_{l'} A_l \tag{5}$$

where $\theta_{ll'}$ describes the mutual statistics of l and l' (Equation 2). Consider the operator

$$(A_l)^{-1}(B_{l'})^{-1} A_l B_{l'}$$

$B_{l'}$ moves l' to the inner edge, then A_l braids l around l'. Then l' goes back to annihilate with its hole and l moves back around. This gives us the mutual statistics of l and l'.

By constructing the algebra of string operators acting on the ground state, we can determine the degeneracy of the ground state. First, we can decompose the action of the operator A_l of quasiparticle $l = (n_1, n_2)$ by moving n_1 many quasiparticles $(1, 0)$ and n_2 many quasiparticles $(0, 1)$ around the annulus. We define $A_1 \equiv A_{(1,0)}$, $A_2 \equiv A_{(0,1)}$. The B string operators are more restrictive - these operators must create quasiparticles that can be annihilated locally at the edge. Since the order parameter W annihilates $l = (2, 2)$ quasiparticles using a local electron spin flip, $l_0 = (2, 2)$ will do the job. We denote this operator B_0. The mutual statistics of l_1, l_2, l_3 then determine the algebra of the string operators:

$$A_1 B_0 = e^{i\pi/2} B_0 A_1, \qquad A_2 B_0 = e^{-i\pi/2} B_0 A_2$$

(A_1 and A_2 commute). Since these operators map the ground state to itself, the ground state must have a degeneracy of at least four. A more careful analysis shows that the degeneracy is exactly four [4]. Since the algebra depends on only the edges and quasiparticle statistics, the groundstate degeneracy is robust against local perturbations. This degeneracy is topologically protected.

In the eigenbasis of B_0, we can write the four ground states as $|n\rangle$, $n \in \mathbf{Z}_4$:

$$B_0|n\rangle = e^{i\pi n/2}|n\rangle$$
$$A_1|n\rangle = |n-1\rangle$$
$$A_2|n\rangle = |n+1\rangle$$

When the annulus has finite width, a time-reversal symmetric tunneling term

$$H_1 \sim (B_0)^2 + h.c.$$

perturbs the energy by ϵ. In the eigenbasis of B_0,

$$H_1|n\rangle \sim e^{i\pi}\epsilon|n\rangle = (-1)^n\epsilon|n\rangle$$

which splits the ground state subspace into a twofold degeneracy. As the width D decreases, the splitting of the levels increases as $\sim e^{-D}$.

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