Quantum Ising criticality on fractal lattices:

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This paper summarizes the work in the literature surrounding solving the Ising model on fractal lattices. The fractional dimensionality offers a novel way to compare the $4-\epsilon$ expansion to numerical results, and the question of whether equations derived in d dimensions can be extrapolated to fractal dimensions in the first place.

Brief Introduction to Fractals:

It has long been understood that the naïve concept of dimensionality can be extended to non-integers.[3] It is tempting to imagine that the dimensionality of a subset of $\mathbb{R}^n$ should simply be n, but there are several problems with this definition: first, it does not allow n to be non-integer, it overlooks the possibility that a subset of $\mathbb{R}^n$ can have dimensionality strictly less than n, and it also forces us to think of the set as being embedded in a higher dimensional space.

To resolve some of these issues, we need a better definition of dimension. An easy way to study dimensionality is to consider how its hypervolume scales as the linear scale is increased. It is natural to say that the hypervolume of an n-dimensional object scales as the linear size raised to the nth power, so this observation motivates the idea that dimensionality should be thought of as a scaling exponent.

In this way the Hausdorff dimension can be defined. We first define the *Hausdorff content*, which provides a notion of hypervolume. If X is a metric space, with some subset S, we can define the Hausdorff content as follows:

$$C_d(S) \equiv \inf \left\{ \sum_i {r_i}^d \;\middle|\; \text{radius } r_i \text{ balls cover } S \right\}. \quad (1)$$

This equation can be interpreted as asking for the minimum amount of d-dimensional volume required to completely cover S. Note that when $d > \dim S$, $C_d(S) = 0$, while when $d < \dim S$, $C_d(S) = \infty$. This allows us to define the Hausdorff dimension via

$$\dim(S) \equiv \inf \left\{ d \mid C_d(S) = 0 \right\}. \quad (2)$$

It is also possible to obtain the Hausdorff dimension by less rigorous means. Consider the Sierpinski Triangle, formed recursively by placing three scaled copies of the previous iteration at the vertices of an equilateral triangle, as shown in Figure 1. Clearly, the fractal can be partitioned into a red, green and blue region, each of which is again a Sierpinski Triangle scaled by a factor 1/2. Thus we conclude that by scaling the Sierpinski triangle by a linear factor of 2, we obtain 3 new copies, so we have

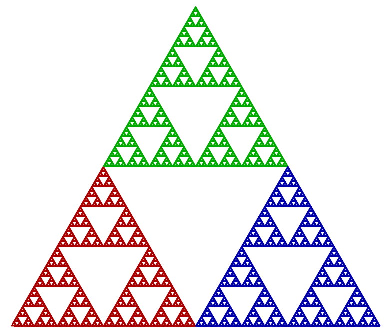

FIG. 1: The Sierpinski Triangle: Each iteration is formed by three copies of the previous iteration, scaled by a factor 1/2, placed to form an equilateral triangle. Note that the Hausdorff content of the Sierpinksi triangle must be three times that of any of the smaller colored Sierpinski Triangles.

$$2^d = 3 \implies d = \log 3/\log 2. \quad (3)$$

It is not hard to see that for a regular n-dimensional simplex, the pattern continues:

$$d = \log(n+1)/\log 2. \quad (4)$$

This has the interesting side-effect that in dimensions one less than a power of 2, the fractal dimension is integer. We will return to this idea later.

Defining an Ising model on the Sierpinski Triangle:[4]

To define an Ising model on the Sierpinski Triangle, we will place a spin $\sigma = \pm 1$ at each vertex. Conceptually, I prefer to imagine that the Sierpinski Triangle is iteratively defined outward, such that the smallest triangle remains the same size, than to imagine it defined iteratively inward. This way, the sample size is infinite, and we have a finite cutoff as in most Real Space RG processes.

Now we proceed to write down the Hamiltonian for the full Sierpinski system; we will write a sum over all the ESTs in the system. First we need to decide how spins will interact with each other. To respect the Sierpinski

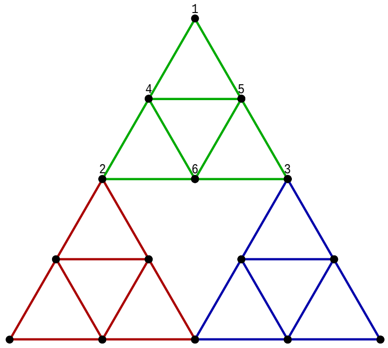

FIG. 2: The Sierpinski setup: We consider the second iteration of the Sierpinski Triangle, made of a red, green, and blue sub-unit. Each colored sub-unit we will call an *elementary Sierpinski Triangle*, or *EST*. We also use the spin index convention from Luscombe et. al.[4]

Geometry, we will consider two spins to be neighbors if and only if they are connected by an edge in the Sierpinski Triangle. In other words, sites that would be adjacent in a triangular lattice are not considered adjacent here if they span across a gap in the ST. Similarly, we also allow for the possibility of a cubic spin term, with odd parity under spin reversal:

$$\sum_{\langle ijk \rangle} \sigma_i \sigma_j \sigma_k, \tag{5}$$

where the sense of $\langle ijk \rangle$ means that triangle ijk is a smallest upward pointing triangle in the ST (such as 145 in figure 2). Again, we exclude downward pointing triangles from having this triple-interaction term because they don't respect the ST geometry. Consequently, we recognize that there are no terms that couple the ESTs! The only way the ESTs talk to each other is through shared points at their vertices (e.g. vertices 2 and 3 of figure 2). For this reason, the Hamiltonian admits a "noninteracting representation":

$$H[\sigma] = \sum_i H_i[\sigma]. \tag{6}$$

Our job is now to figure out how to write $H_i[\sigma]$, which can come from the standard nearest neighbor interactions, from the external magnetic field, and finally through our strange triple-interaction:

$$\begin{aligned} H_i[\sigma] = K\big(&\underbrace{\sigma_1(\sigma_4+\sigma_5)+\sigma_2(\sigma_4+\sigma_6)}_{\text{pairwise interactions via 1 and 2}} \\ &+\underbrace{(\sigma_3+\sigma_4)(\sigma_5+\sigma_6)}_{\text{diamond 4536}}+\underbrace{\sigma_5\sigma_6}_{\text{5 and 6}}\big) \\ &+\underbrace{\frac{B}{2}(\sigma_1+\sigma_2+\sigma_3)}_{\text{vertex magnetic field}}+\underbrace{B(\sigma_1+\sigma_2+\sigma_3)}_{\text{non-vertex magnetic field}} \\ &+\underbrace{K_3(\sigma_1\sigma_4\sigma_5+\sigma_2\sigma_4\sigma_6+\sigma_3\sigma_5\sigma_6)}_{\text{triple-interactions}}. \end{aligned} \tag{7}$$

Notice in the first magnetic field term, there is a factor of $\frac{1}{2}$ to prevent the magnetic field term on the vertices of the ESTs from being double-counted by their neighbors.

Now we must sum over the spins $\{4, 5, 6\}$, and identify the decimated system with the original one:

$$\exp H[\sigma'] = \sum_{\sigma}{}' \exp H[\sigma], \tag{8}$$

where the σ is summed to respect σ'. After setting the external field and the triple-interaction to zero, the recursion relation becomes[4]

$$e^{4K'} = e^{4K}\left(\frac{1-e^{-4K}+4e^{-8K}}{1+3e^{-4K}}\right). \tag{9}$$

This recursion relation has a fixed point at $k = 0$ and $k = \infty$, with no finite temperature phase transition. Because of the finite (3) ramification of the ST, perhaps this has something to do with the lack of interesting phase transitions, in much the same way that the 1D Ising model does not have a finite temperature phase transition (itself having ramification number 2).

Quantum Ising Model:[1]

We now turn to the Quantum Ising model on the Sierpinski Triangle, but this time we use a slight modification of the Sierpinski Lattice; the iteration step connects three copies of the previous iteration via edges, rather than merely identifying the vertices. This is a technical simplification that does not change the geometric resemblance of the lattice to a ST.

Goals:

The big question here is how seriously we should take the notion of a fractal dimension, with regards to comparing with analytical work via ϵ-expansion. In other words, in one sense the ST is a 2D lattice because it is a planar graph, but on the other hand it is tempting to associate its dimension with the fractal dimension of $\frac{\log 3}{\log 2} \approx 1.585$. Let us take the second interpretation, and suppose that the ST serves as a kind of extrapolation to dimension between 1 and 2. Unfortunately, there is few success in the literature on fractal lattices that can be readily solved numerically that reveal critical phenomena.[5] It appears that this disappointment is due to the fact that many such fractal lattices have finite ramification number, meaning arbitrarily large chunks can be "cut-out" of the fractal (rendering them disconnected with the rest) by only cutting finitely many edges. However, what is usually possible is that studying fractal lattices can signal existence or non-existence of quantum criticality in higher dimensions. It is for this reason that Yoshida and Kubica call fractal lattices a "probe" for higher dimensional physics.[1]

Quantum Sierpinski Triangle:

The Quantum Sierpinski Triangle has been studied in a transverse magnetic field with the following Hamiltonian:[1]

$$H(\epsilon) = -(1-\epsilon)\sum_{(i,j)\in E} Z_i Z_j - \epsilon \sum_{j\in V} X_j. \tag{10}$$

This Hamiltonian allows us to linearly interpolate between the nearest-neighbor interactions and the effect of the transverse field, and the sums run over edges and vertices respectively. Via the Quantum-Classical Correspondence, the classically equivalent system is the "Trotterized" Sierpinski Triangle made from a stack of coupled ST lattices. Then the problem can be solved via Monte-Carlo methods.[1] Yoshida and Kubica did this simulation and found the following results with a few assumptions:

1. The Quantum system is at a fixed RG point
2. The correlation length diverges to the size of the lattice L.

They were able to derive the following critical exponents via numerical methods which are beyond the scope of this paper:

$$\alpha = 0.034, \nu = 0.76, \frac{\beta}{\nu} = 0.237, \frac{\gamma}{\nu} = 2.111.$$

It is interesting to note that it is possible to solve for the dimension of the system using the scaling relation

$$\nu d = 2 - \alpha \implies d \approx \frac{2 - 0.034}{0.76} = 2.587. \tag{11}$$

Recalling that the dimension of the Trotterized version of the Sierpinski Triangle is one larger than the dimension of the standard ST, subtracting 1 from the result from eq.(11), we obtain 1.587, which is an incredible 0.1% error.

Quantum Sierpinski Tetrahedron:[1]

Yoshida and Kubica also point out an interesting fact I proved in eq. (4): the fractal dimension of the Sierpinski Tetrahedron (STet), is exactly 2. It is a truth universally acknowledged that a universality class in possession of a quantum phase transition must be determined fully by its symmetry class and the dimensionality. Accordingly, this coincidence of a fractal and non-fractal dimension provides an intriguing way to test this claim. When Yoshida and Kubica did the numerics on the STet, they discovered the following critical exponents:

$$\nu = 0.0660,\ \gamma = 1.5800, \tag{12}$$

Whereas the critical exponents for the 2D Quantum Ising model are

$$\nu = 0.06301,\ \gamma = 1.2372. \tag{13}$$

While ν is pretty close, γ (as well as other critical exponents), doesn't do quite as well. This is evidence that the universality class is not entirely determined by the dimension and symmetry group of the lattice. For reasons I do not understand, this result may shed some light on conformal bootstrapping.[1]

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[6] A. Kubica and B. Yoshida, arXiv:1402.0619.